Two results on the essential self-adjointness of the Schrödinger operator

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Master's thesis

29.09.2020

HELSINGIN YLIOPISTO — HELSINGFORS UNIVERSITET — UNIVERSITY OF HELSINKI

Tiedekunta/Osasto — Fakultet/Sektion — Faculty	Laitos — Institution — Department
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Tekijä — Författare — Author

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Työn nimi — Arbetets titel — Title

Two results on the essential self-adjointness of the Schrödinger operator

Oppiaine — Läroämne — Subject

Mathematics

Työn laji — Arbetets art — Level	Aika — Datum — Month and year	Sivumäärä — Sidoantal — Number of pages
Master's thesis	September 2020	52

Tiivistelmä — Referat — Abstract

Spectral theory is a powerful tool when applied to differential equations. The fundamental result being the spectral theorem of Jon Von Neumann, which allows us to define the exponential of an unbounded operator, provided that the operator in question is self-adjoint. The problem we are considering in this thesis, is the self-adjointness of the Schrödinger operator $T = -\Delta + V$, a linear second-order partial differential operator that is fundamental to non-relativistic quantum mechanics. Here, Δ is the Laplacian and V is some function that acts as a multiplication operator. We will study T as a map from the Hilbert space $H = L^2(\mathbb{R}^d)$ to itself. In the case of unbounded operators, we are forced to restrict them to some suitable subspace. This is a common limitation when dealing with differential operators such as T and the choice of the domain will usually play an important role.

Our aim is to prove two theorems on the essential self-adjointness of T, both originally proven by Tosio Kato.

We will start with some necessary notation fixing and other preliminaries in chapter 2. In chapter 3 basic concepts and theorems on operators in Hilbert spaces are presented, most importantly we will introduce some characterisations of self-adjointness.

In chapter 4 we construct the test function space $D(\Omega)$ and introduce distributions, which are continuous linear functionals on $D(\Omega)$. These are needed as the domain for the adjoint of a differential operator can often be expressed as a subspace of the space of distributions.

In chapter 5 we will show that T is essentially self-adjoint on compactly supported smooth functions when $d = 3$ and V is a sum consisting of an L^2 term and a bounded term. This result is an application of the Kato-Rellich theorem which pertains to operators of the form $A + B$, where B is bounded by A in a suitable way. Here we will also need some results from Fourier analysis that will be revised briefly.

In chapter 6 we introduce some mollification methods and prove Kato's distributional inequality, which is important in the proof of the main theorem in the final chapter and other results of similar nature.

The main result of this thesis, presented in chapter 7, is a theorem originally conjectured by Barry Simon which says that T is essentially self-adjoint on $C_c^\infty(\mathbb{R}^d)$, when V is a non-negative locally square integrable function and d is an arbitrary positive integer. The proof is based around mollification methods and the distributional inequality proven in the previous chapter.

This last result, although fairly unphysical, is somewhat striking in the sense that usually for T to be (essentially) self-adjoint, the dimension d restricts the integrability properties of V significantly.

Avainsanat — Nyckelord — Keywords

Hilbert spaces, Schrödinger operators, Self-adjointness, Distribution theory, Kato's inequality

Säilytyspaikka — Förvaringsställe — Where deposited

Kumpulan tiedekirjasto

Muita tietoja — Övriga uppgifter — Additional information

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Chapter 1

Introduction

One of the fundamental results in functional analysis is the spectral theorem, which among other things, allows us to construct the functional calculus of self-adjoint operators. That is, we can define $f(A)$ where f is a suitably regular function and A is a self-adjoint operator on a Hilbert space. In particular it is possible to give rigorous meaning to the exponential $U(t) = e^{itA}$ for an unbounded operator A, provided that it is self-adjoint. This is useful since usually partial differential operators are not bounded, but sometimes it is possible to show that they are at least essentially self-adjoint, that is, they have a unique self-adjoint extension. When A is an operator corresponding to some partial differential equation, the exponential can then be used to find solutions. The unitary group formed by $U(t)$ with $t \in \mathbb{R}$ is also fundamental to quantum physics. A thorough presentation of the above concepts is given in [2].

In this thesis we consider the self-adjointness of the Schrödinger operator

$$T = -\Delta + V.$$

This is one of the main problems that arises in non-relativistic quantum mechanics alongside analysis of the spectrum of T and its scattering properties.

The bulk of this text will be devoted to building the necessary tools to handle two theorems, both of which are originally due to T. Kato. We wish to find suitable criteria which guarantee that T is essentially self-adjoint on the domain of compactly supported smooth functions. Usually these types of problems are heavily dependent on the dimension of the underlying Euclidean space. This is the case for our first result for $V = V_1 + V_2$ where $V_1 \in L^2(\mathbb{R}^3)$ and $V_2 \in L^\infty(\mathbb{R}^3)$.

The second result is not very relevant physically, but it is rather surprising since we place no restriction on the dimension. Instead we assume that V is non-negative and locally square integrable. This theorem was conjectured by B. Simon in his 1973 paper [8] and first proven by T. Kato in [4]. The proof utilizes a distributional inequality also due to Kato, which is useful in other contexts as well. Our main sources will be [2] and [3], the first two volumes of the seminal book series on mathematical physics by Reed and Simon.

After some preliminaries are established, we will start by going through the basics of Hilbert space theory for unbounded operators in chapter 3. Also some characterization results are presented for self-adjoint and essentially self-adjoint operators. We will briefly define the spectrum for unbounded operators and prove that it is real in the self-adjoint case.

Along the way some tools from distribution theory will be needed, so a brief introduction is presented in chapter 4 with some proofs omitted. We will start from the concept of a topological vector space and follow the constructions presented by W. Rudin in his classic text [5].

In chapter 5 we prove the Kato-Rellich theorem and apply it to the operator T in three dimensions with the potential $V = V_1 + V_2$ mentioned earlier. This will require some standard results from Fourier analysis which will be provided with references to the proofs.

In chapter 6 we introduce some mollification methods for distributions and L^p spaces. These will then be used alongside rest of the developed machinery to prove Kato's inequality and finally our self-adjointness result for positive potentials in the final chapter.

Chapter 2

Preliminaries

Throughout this text it is assumed that the reader is familiar with the basic notions and results in point-set topology, functional analysis in Banach spaces and elementary real analysis as presented in a standard graduate level introductory course to the subjects. Some basic results of Fourier analysis will be needed as well, but the reader will be provided a source for these as needed.

Next we will fix some basic definitions and notations.

Definition 2.1. *Let X and Y be vector spaces and let T be map from X to Y. Then the kernel and range of T are defined as*

$$Ker(T) = \{x \in X : Tx = 0\}$$

and

$$Ran(T) = \{y \in Y : y = Tx \text{ for some } x \in X\}$$

respectively.

For $z \in \mathbb{C}$, $\overline{z}$ denotes the complex conjugate, and $\Re(z)$ and $\Im(z)$ denote the real and imaginary parts of z respectively. For a set V in a topological space X, $\overline{V}$ denotes the closure, ∂V denotes the boundary and $\mathbf{int}V$ denotes the interior of V. In a metric space $B(x, r)$ will denote a ball centered at x with radius r.

It is assumed that the reader is familiar with the concept of L^p spaces, but we will also encounter locally integrable functions which we define as follows.

Definition 2.2. *For an open set Ω in $\mathbb{R}^d$ we denote by $L^p_{loc}(\Omega)$ the set of Lebesgue measurable functions f for which*

$$\int_K |f|^p dx < \infty$$

for every compact set $K \in \Omega$.

We denote the set of natural numbers with 0 excluded by $\mathbb{N}$ and define $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$. We will use the standard multi-index notation i.e., for a natural number d, multi-index

$$\alpha = (\alpha_1, \alpha_2, ..., \alpha_d) \in \mathbb{N}^d$$

and a vector

$$x = (x_1, x_2, ..., x_d)$$

in $\mathbb{R}^d$ or $\mathbb{C}^d$, we denote

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} ... x_d^{\alpha_d}$$

and

$$|\alpha| = \sum_{k=1}^{d} \alpha_k.$$

Similarly, we denote the n-fold partial derivative in the k:th component by D_k^n and denote

$$D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} ... D_d^{\alpha_d}.$$

Definition 2.3. *Let $\mathbb{K}$ denote either of the fields $\mathbb{R}$ or $\mathbb{C}$. Let $\Omega \subset \mathbb{R}^n$ be an open set. We denote the space of continuous functions from Ω to $\mathbb{K}$ by $C(\Omega)$. Further, we fix the following definitions:*

1. *for $k \in \mathbb{N}$ we denote $C^k(\Omega) = \{\phi \in C(\Omega) : \exists D^\alpha \phi \in C(\Omega)$ for all $|\alpha| \leq k\}$,*
2. *$C^\infty(\Omega) = \{\phi \in C(\Omega) : \phi \in C^k(\Omega)$ for all $k \in \mathbb{N}\}$,*
3. *for $\beta \in \mathbb{N}_0 \cup \{\infty\}$ we denote by $C_c^\beta(\Omega)$ the set of functions ϕ in $C^\beta(\Omega)$ whose support lies in some compact set $K \subset \Omega$.*

For our purposes $\mathbb{K}$ is usually the field of complex numbers. Unless otherwise stated functions in $L^p(\mathbb{R}^d)$ are assumed to be complex-valued.

We take as known that $C_c^\infty(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$ for all $p < \infty$. The proof can be found from theorem 9.6 and corollary 9.7 of [10] for example.

In a complex inner product space X we assume the convention that the inner product $(\cdot|\cdot)$ be conjugate linear in the second parameter, that is, $(u|zv) = \overline{z}(u|v)$ for $u, v \in X$ and $z \in \mathbb{C}$.

Chapter 3

Operator theory

3.1 Operators on Hilbert spaces

We will now go through some basic aspects of unbounded linear operators in Hilbert spaces. We do not usually require our operators to be bounded since this is quite a strict condition especially when dealing with differential operators.

Definition 3.1. *An inner product space $(H, (\cdot|\cdot))$ is said to be a Hilbert space if it is complete with respect to the norm $\|x\|_H = \sqrt{(x|x)}$ induced by the inner product.*

We omit the subscript in the norm symbol indicating the space if it is obvious from context.

For a general linear operator A the domain of definition plays a key role in determining what properties the operator has. Thus we adopt the following standard notation.

Notation. If H is a Hilbert space and A is a linear operator $A : D \to H$ where $D \subset H$ we say that A is an operator on H with domain $D(A) := D$.

Clearly for A above to be linear its domain must be a linear subspace. From now on H denotes an infinite dimensional complex Hilbert space and operators are assumed to be linear unless otherwise stated.

Definition 3.2. *Let A be an operator on a Hilbert space H. We assert the following definitions.*

1. *If $D(A)$ is a dense subspace of H then A is said to be densely defined.*

2. *The graph of A is denoted by $\mathcal{G}(A)$ where*

$$\mathcal{G}(A) := \{(x, y) \in H \times H : x \in D(A) \text{ and } y = Ax\}.$$

3. *The operator A is said to be closed if $\mathcal{G}(A)$ is a closed subspace of $H \times H$ with respect to the graph norm*

$$\|(x,y)\|_{H\times H} := \sqrt{\|x\|_H^2 + \|y\|_H^2}.$$

4. *An operator B on H is an extension of A if $D(A) \subset D(B)$ and $B|_{D(A)} = A$. In this case we adopt the notation $A \subset B$.*

5. *If the operator A has a closed extension then A is said to be closable.*

Example 3.3. The operator

$$A : C^1([0,1]) \to L^2([0,1])$$

defined as $Au(t) = \frac{d}{dt}u(t)$ is densely defined and unbounded. For let $f_n(t) = e^{int}$ where $n \in \mathbb{N}$. Then

$$\|f_n\|_{L^2} = \Big(\int_0^1 |e^{int}|^2 dt\Big)^{\frac{1}{2}} = 1$$

and

$$\|Af_n\| = \|\frac{d}{dt}f_n\| = \Big(\int_0^1 |ine^{int}|^2 dt\Big)^{\frac{1}{2}} = n.$$

So there is no constant $c > 0$ for which $\|Au\|_{L^2([0,1])} \leq c\|u\|_{L^2([0,1])}$ for all $u \in C^1([0,1])$.

Notice that the graph norm comes from the inner product

$$((u,v)|(w,y))_{H\times H} = (u|w)_H + (v|y)_H$$

making $H \times H$ also a Hilbert space.

The following lemma shows that a closable operator A has a minimal closed extension which we will call the closure of A.

Lemma 3.4. *Operator A on a Hilbert space H is closable if and only if $\overline{\mathcal{G}(A)}$ is the graph of some closed operator B on H.*

Proof. First assume that $\mathcal{G}(B) = \overline{\mathcal{G}(A)}$ for some closed operator B. Now $\mathcal{G}(A) \subset \mathcal{G}(B)$ so we have $(x, Bx) = (x, Ax)$ for all $x \in D(A) \subset D(B)$ and thus B extends A.

Assume then that there exists a closed extension A' of A. Since both $\mathcal{G}(A')$ and $\overline{\mathcal{G}(A)}$ are closed in $H \times H$ and they both contain the graph of A we have $\overline{\mathcal{G}(A)} \subset \mathcal{G}(A')$. Define for $j \in \{1,2\}$ the projection operators $P_j : H \times H \to H$ where $P_j(x_1, x_2) = x_j$. Let

$$D(B) = Ran(P_1|_{\overline{\mathcal{G}(A)}})$$

and define a map $B : D(B) \to H$ such that $B := P_2 G$ where $G : D(B) \to \overline{\mathcal{G}(A)}$ is the linear map $Gx = (x, A'x)$.

Now the operator B is linear on the subspace $Ran(P_1|_{\overline{\mathcal{G}(A)}})$ containing $D(A)$ since P_2 and G are linear. It also extends A because $Bx = P_2 Gx = P_2(x, A'x) = A'x = Ax$ whenever $x \in D(A)$. Also the graph of B is exactly $\overline{\mathcal{G}(A)}$. □

This result motivates the notation $\overline{A} = B$ when A and B are as above in the sense that B is the smallest closed extension of A regarding domains.

Notice that in the above proof we only used the fact that H is a normed vector space so the lemma actually holds in a more general setting than stated.

Example 3.5. If T is a densely defined bounded operator on a Hilbert space H, it can be extended by the B.L.T. theorem[1] to the whole space preserving boundedness. If $\tilde{T}$ is the mentioned extension and $(x_n, \tilde{T}(x_n))$ is a converging sequence in $H \times H$ then $x_n \to x$ in H and by continuity $\tilde{T}(x_n) \to \tilde{T}(x)$ in H. Thus every bounded and densely defined operator is closable and the closure will be bounded as well.

3.2 Adjoint of an operator

We begin by defining the adjoint of a general linear operator on a Hilbert space.

Definition 3.6. *Let $A : D(A) \to H$ be a densely defined linear operator on a Hilbert space H. Denote*

$$D(A^*) = \{v \in H : \exists u \in H \text{ such that } (Aw|v) = (w|u) \text{ for all } w \in D(A)\}.$$

The adjoint of A is the operator

$$A^* : D(A^*) \to H$$

with $A^*v := u$ where u is the element given by the definition of $D(A^*)$.

From the definition we see that for all $v \in D(A^*)$ the functional $(A \cdot |v)$ is bounded since $v \in D(A^*)$ implies

$$|(Aw|v)| \leq \|w\|\|u\|$$

for all $w \in D(A)$ with $\|u\|$ as a constant. Since the functional $(A \cdot |v)$ is bounded and $D(A)$ is dense we can uniquely extend it to the whole space by the B.L.T. theorem. Then the Riesz representation theorem[2] guarantees that the element $A^*v = u$ is unique. Thus

[1]Here B.L.T. is short for bounded linear transform. See theorem I.7 of [2].

[2]Note that there are many related results in different contexts also called the Riesz representation theorem. For the proof of this one, see theorem II.4 of [2]

A^* is well defined. That the adjoint of a linear operator is also linear, follows from the properties of the inner product.

Now we will present some basic properties of the adjoint. First let us consider the linear isometry $V : H \times H \to H \times H$ defined as

$$V(u, v) = (v, -u).$$

For a metric space (X, d_X) a map $f : X \to X$ is called an isometry if it preserves the metric

$$d_X(F(x), F(y)) = d_X(x, y)$$

for all $x, y \in X$. This is clearly true for the map V here since it is linear and it preserves the inner product and thus the norm.

Now the graph of the adjoint can be represented in terms of the map V and the graph of the original operator.

Lemma 3.7. *Let A be an operator on a Hilbert space H. Then*

$$\mathcal{G}(A^*) = \left[V\left(\overline{\mathcal{G}(A)}\right)\right]^{\perp}.$$

Proof. First we will show that

$$\left[V\left(\overline{\mathcal{G}(A)}\right)\right]^{\perp} = \left[\overline{V\left(\mathcal{G}(A)\right)}\right]^{\perp} = \left[V\left(\mathcal{G}(A)\right)\right]^{\perp}. \tag{3.1}$$

The last equality is just the fact that the orthogonal complement of a set and its closure are the same. This follows from the continuity of the inner product.

For the first equality let $F : X \to X$ be an isometry in a complete metric space X and $A \subset X$. Then for $y \in F(\overline{A})$ we can find an element $x \in \overline{A}$ such that $F(x) = y$ and a sequence $(x_k)_{k=1}^{\infty} \subset A$ which converges to x. Since F is an isometry $d_X(F(x), F(x_k)) = d_X(x, x_k)$ and so the $F(x_k)$ converge to $F(x)$. Thus $F(\overline{A}) \subset \overline{F(A)}$.

Similarly for $y \in \overline{F(A)}$ there exists a sequence $(F(x_k))_{k=1}^{\infty} \subset F(A)$ where $(x_k)_{k=1}^{\infty} \subset A$ and that converges to y. Now $(F(x_k))_{k=1}^{\infty}$ is a Cauchy sequence which means that also $(x_k)_{k=1}^{\infty}$ must be Cauchy. By completeness $x_k \to x$ for some $x \in \overline{A}$ and by continuity $F(x) = y$. This proves (3.1).

Next recall that the inner product of $H \times H$ is defined as

$$((u_1, u_2)|(v_1, v_2))_{H \times H} = (u_1|v_1)_H + (u_2|v_2)_H$$

for $(u_1, u_2), (v_1, v_2) \in H \times H$. Thus for $u \in D(A)$ we get

$$(V(u, Au)|(v, w)) = (Au|v) - (u|w). \tag{3.2}$$

by the definition of V. Now if $v \in D(A^*)$ and $w = A^*v$ the right-hand side of (3.2) will be zero for all $u \in D(A)$ since

$$(u|w) = (u|A^*v) = (Au|v)$$

by definition and so $(v, A^*v) \in \big[V\big(\mathcal{G}(A)\big)\big]^\perp$. On the other hand if $(v, w) \in \big[V\big(\mathcal{G}(A)\big)\big]^\perp$ then

$$(V(u, Au)|(v, w)) = 0$$

for all $u \in D(A)$ and so $w = A^*v$. Thus $(v, w) \in \mathcal{G}(A^*)$. With (3.1) this proves the lemma. □

Theorem 3.8. *Let A be an operator on H. Then the following properties hold.*

1. *A^* is a closed operator.*
2. *If A is closable then A^* is densely defined and $(A^*)^* = \overline{A}$. We will use the notation A^{**} for $(A^*)^*$.*
3. *If A is closable then $(\overline{A})^* = A^*$.*
4. *If A^* is densely defined then A is a closable operator.*

Proof. 1. Let $(v_k, w_k)_{k=1}^\infty \subset \mathcal{G}(A^*)$ be a convergent sequence in $H \times H$ with (v, w) as a limit. Then for an arbitrary $u \in D(A)$ we have

$$(Au|v) = \lim_{k\to\infty} (Au|v_k) = \lim_{k\to\infty} (u|A^*v_k) = \lim_{k\to\infty} (u|w_k) = (u|w).$$

Thus $v \in D(A^*)$ and $w = A^*v$ and so A^* is a closed operator.

2. Assume that A is closable and $D(A^*)$ is not dense. Since there is a non-zero element $u \in H \setminus \overline{D(A^*)}$ and naturally $(u|0) = 0$, there must also exist a non-zero element w in $D(A^*)^\perp = (\overline{D(A^*)})^\perp$. Otherwise $u \in (D(A^*)^\perp)^\perp = \overline{D(A^*)}$, which would be a contradiction.

Now for all $v \in D(A^*)$ we have

$$((0, w)|V(v, A^*v)) = ((0, w)|(A^*v, -v))_{H\times H} = (0|A^*v) - (w|v) = 0$$

and thus $(0, w) \in \big[V\big(\mathcal{G}(A^*)\big)\big]^\perp$. From lemma 3.7 and its proof we get $\mathcal{G}(A^*)^\perp = V(\overline{\mathcal{G}(A)})$. Since $V^2 = -I$ it follows that $V^2(S) = S$ for any subspace S of $H \times H$ and so

$$\overline{\mathcal{G}(A)} = V(\mathcal{G}(A^*)^\perp). \tag{3.3}$$

Notice that $\mathcal{G}(A^*)$ is a closed subspace of $H\times H$ by property 1. which we just proved. From the definition of V we see that

$$(V(x,y)|(v,w))_{H\times H} = ((x,y)|V(v,w))_{H\times H}$$

holds for all $x,y,v,w \in H$. If M is a closed subspace of $H \times H$, then we have the following equivalencies

$$\begin{aligned}(x,y) \in V(M^\perp) &\Leftrightarrow (-y,x) \in M^\perp \\ &\Leftrightarrow -V(x,y) \in M^\perp \\ &\Leftrightarrow \forall (v,w) \in M : (V(x,y)|(v,w)) = 0 \\ &\Leftrightarrow \forall (v,w) \in M : ((x,y)|V(v,w)) = 0 \\ &\Leftrightarrow (x,y) \in V(M)^\perp\end{aligned}$$

and so $V(M^\perp) = V(M)^\perp$.

We have thus shown that $\overline{\mathcal{G}(A)} = \left[V(\mathcal{G}(A^*))\right]^\perp$ and since A was closable $\overline{\mathcal{G}(A)} = \mathcal{G}(\overline{A})$ by lemma 3.4. So now $(0,w) \in \mathcal{G}(\overline{A})$ which means that $w = \overline{A}0 = 0$. This is a contradiction and so $D(A^*)$ must be dense.

Now the operator A^{**} exists since A^* was densely defined. Again we utilize lemma 3.7 to get

$$\overline{\mathcal{G}(A)} = \left[V(\mathcal{G}(A^*))\right]^\perp = \left[V\left(\overline{\mathcal{G}(A^*)}\right)\right]^\perp = \mathcal{G}(A^{**})$$

which shows that indeed $A^{**} = \overline{A}$, when A is closable.

3. By the previous results when A is closable we have

$$A^* = \overline{A^*} = (A^*)^{**} = (A^{**})^* = (\overline{A})^*.$$

4. Since A^* is densely defined, A^{**} is well defined. As A^* is also closed, from the previous proofs we get

$$\mathcal{G}(A^{**}) = \left[V\left(\overline{\mathcal{G}(A^{***})}\right)\right]^\perp = \left[V\left(\overline{\mathcal{G}(\overline{A^*})}\right)\right]^\perp = \left[V\left(\overline{\mathcal{G}(A^*)}\right)\right]^\perp = \overline{\mathcal{G}(A)}.$$

This shows that A is closable by lemma 3.4, since A^{**} is closed.

□

In operator theory and applications the situation where $A = A^*$ holds, is of special interest. In general this usually is a bit too much to ask, but in some cases we can still find a unique extension that has this property and which is almost the same as A. To make this precise we give the following definition.

Definition 3.9. *Let A be an operator on a Hilbert space H. Then*

1. *A is said to be symmetric if the adjoint A^* extends A,*
2. *self-adjoint if A and A^* are equal,*
3. *essentially self-adjoint if the closure $\overline{A}$ exists and is self-adjoint.*

Immediate consequence of theorem 3.8 is that a symmetric operator is always closable. It turns out that the closure of a symmetric operator is also symmetric.

Lemma 3.10. *Let A be a symmetric operator on a Hilbert space H. Then it is closable and its closure is also symmetric.*

Proof. As mentioned above closability of A is a clear consequence of theorem 3.8.

We also have

$$\mathcal{G}(A) \subset \overline{\mathcal{G}(A)} = \mathcal{G}(\overline{A})$$

and again by the theorem 3.8 the set $\mathcal{G}(A^*)$ is closed so it must be that $\mathcal{G}(\overline{A}) \subset \mathcal{G}(A^*)$. So $\overline{A}$ is also symmetric. □

Example 3.11. The Laplacian operator, defined as

$$\Delta := \sum_{k=1}^{d} D_k^2$$

is symmetric in $L^2(\mathbb{R}^d)$ with the domain $C_c^\infty(\mathbb{R}^d)$. This follows directly from Green's first identity[3]. First let $\phi, \psi \in C_c^\infty(\mathbb{R}^d)$ be real valued and choose $R > 0$ such that $B := B(0, R)$ contains the supports of both ϕ and ψ. Then

$$\int_{\mathbb{R}^d} \phi\Delta\psi + \nabla\phi \cdot \nabla\psi dx = \int_B \phi\Delta\psi + \nabla\phi \cdot \nabla\psi dx = \oint_{\partial B} \phi(\nabla\psi \cdot \mathbf{n})dS = 0$$

where the second equality is the Green's identity. Similarly

$$\int_{\mathbb{R}^d} \psi\Delta\phi + \nabla\psi \cdot \nabla\phi dx = 0.$$

Since the dot product is commutative we have

$$(\phi, \Delta\psi) + \int_{\mathbb{R}^d} \nabla\phi \cdot \nabla\psi dx = 0 = (\Delta\phi, \psi) + \int_{\mathbb{R}^d} \nabla\phi \cdot \nabla\psi dx.$$

[3] An immediate consequence of the divergence theorem, the proof of which is quite messy. It can be found from chapter 10 of [6].

This proves the claim for real valued functions in $C_c^\infty(\mathbb{R}^d)$.

If ϕ and ψ are allowed to be complex valued we have

$$\Delta\phi\overline{\psi} = \Delta\Re(\phi)\Re(\psi) - i\Delta\Re(\phi)\Im(\psi) + i\Delta\Im(\phi)\Re(\psi) + \Delta\Im(\phi)\Im(\psi). \tag{3.4}$$

Thus if we integrate both sides of (3.4), we can use the real version on each of the terms on the right-hand side to get

$$(\Delta\phi, \psi) = \int_{\mathbb{R}^d} \Delta\Re(\phi)\Re(\psi)dx - i\int_{\mathbb{R}^d} \Delta\Re(\phi)\Im(\psi)dx + i\int_{\mathbb{R}^d} \Delta\Im(\phi)\Re(\psi)dx + \int_{\mathbb{R}^d} \Delta\Im(\phi)\Im(\psi)dx$$

$$= \int_{\mathbb{R}^d} \Re(\phi)\Delta\Re(\psi)dx - i\int_{\mathbb{R}^d} \Re(\phi)\Delta\Im(\psi)dx + i\int_{\mathbb{R}^d} \Im(\phi)\Delta\Re(\psi)dx + \int_{\mathbb{R}^d} \Im(\phi)\Delta\Im(\psi)dx$$

$$= \int_{\mathbb{R}^d} \phi\Delta\overline{\psi}dx = (\phi, \Delta\psi).$$

Example 3.12. If f is a real valued measurable function on $\mathbb{R}^d$, then the multiplication operator $M_f : D(M_f) \to L^2(\mathbb{R}^d)$ defined as $M_f\phi = f\phi$, is self-adjoint when

$$D(M_f) = \{\phi \in L^2(\mathbb{R}^d) : f\phi \in L^2(\mathbb{R}^d)\}.$$

Let $\phi \in L^2(\mathbb{R}^d)$ be arbitrary. Then $|\frac{\phi}{|f|^2+1}| \leq |\phi|$ and

$$\frac{f\phi}{|f|^2+1} \in L^2(\mathbb{R}^d)$$

since $\left|\frac{f}{|f|^2+1}\right| \leq 1$ and thus

$$\frac{\phi}{|f|^2+1} \in D(M_f).$$

Now if $\phi \in D(M_f)^\perp$ we have

$$0 = \left(\phi \Big| \frac{\phi}{|f|^2+1}\right) = \int_{\mathbb{R}^d} |\phi|^2 \frac{1}{|f|^2+1} dx$$

and thus $\phi = 0$ almost everywhere. Hence $D(M_f)$ is dense.

Now let $\phi \in D(M_f^*)$ and $\psi \in L^2(\mathbb{R}^d)$. Then

$$\left(\frac{f\psi}{|f|^2+1}\Big|\phi\right) = \left(\frac{\psi}{|f|^2+1}\Big|M_f^*\phi\right) \Longleftrightarrow \left(\psi\Big|\frac{f\phi}{|f|^2+1}\right) = \left(\psi\Big|\frac{M_f^*\phi}{|f|^2+1}\right)$$

since f was real valued. Now we have

$$\frac{f\phi}{|f|^2+1} = \frac{M_f^*\psi}{|f|^2+1} \Longrightarrow f\phi = M_f^*\phi$$

and so M_f is self-adjoint.

In general a symmetric operator A can have multiple self-adjoint extensions or none at all[4], but essential self-adjointness guarantees that $\overline{A}$ is the only self-adjoint extension of A.

To see this, let A and B be symmetric operators such that $A \subset B$. If $u \in B(A^*)$ and $v \in D(A)$, then $Av = Bv$ and

$$(Av|u) = (Bv|u) = (v|B^*u),$$

which implies $B^* \subset A^*$. So taking adjoints inverts the inclusion. Now, if A is essentially self-adjoint and B is some self-adjoint extension of A, we have

$$A \subset B \Longrightarrow B \subset A^* \Longrightarrow A^{**} \subset B \Longrightarrow B \subset A^{***}.$$

Since $A^{**} = \overline{A} = (\overline{A})^* = A^{***}$ the uniqueness follows.

It will be good to keep in mind that if A is symmetric, then $A \subset \overline{A} = A^{**} \subset A^*$ and if it is essentially self-adjoint, $A \subset \overline{A} = A^* = A^{**}$. For self-adjoint operators we have equalities throughout.

3.3 Characterizations of self-adjointness

Now we will introduce some elementary criteria for self-adjointness that will be useful later on. First a simple but effective decomposition result.

Lemma 3.13. *If A is a densely defined linear operator on a Hilbert space H, then the space admits the following orthogonal decomposition*

$$H = Ker(A^*) \oplus \overline{Ran(A)}.$$

Proof. Clearly $Ker(A^*)$ is a subspace of H since A^* is linear. It is also closed. To see this let $\{u_k\}_{k=1}^{\infty}$ be a sequence in $Ker(A^*)$ converging to some u in H. Now

$$(Av|u) = \lim_{k\to\infty}(Av|u_k) = \lim_{k\to\infty}(v|A^*u_k) = (v|0)$$

for all $v \in D(A)$. By the definition of the adjoint $u \in D(A^*)$ and $A^*u = 0$, thus making $Ker(A^*)$ closed. By elementary functional analysis[5] we have the decomposition

$$H = Ker(A^*) \oplus Ker(A^*)^{\perp}.$$

[4] See section VIII.2 of [2].

[5] See theorem II.3 of [2].

We will show that $Ker(A^*) = Ran(A)^\perp$, which will give

$$Ker(A^*)^\perp = (Ran(A)^\perp)^\perp = \overline{Ran(A)}.$$

Firstly, let $w \in Ran(A)^\perp$. Now

$$(0|u) = 0 = (w|Au)$$

for all $u \in D(A)$. Thus $w \in D(A^*)$ and $A^* w = 0$ by definition.

Next, let $w \in Ker(A^*)$ and $v \in D(A)$. Then we have $(w|Av) = (A^* w|v) = 0$ and so $w \in Ran(A)^\perp$. □

Theorem 3.14. *Let A be a densely defined symmetric operator on a Hilbert space H. Then the following conditions are equivalent.*

1. *A is self-adjoint.*
2. *A is closed and $Ker(A^* \pm iI) = \{0\}$.*
3. *The range of $A \pm iI$ is equal to H.*

Proof. *1 ⇒ 2* : By theorem 3.8

$$A = A^* = A^{**} = \overline{A}$$

so A is closed. Let $u \in Ker(A^* + iI)$. Then

$$i(u|u) = (-A^* u|u) = (-u|Au) = (u| - A^* u) = (u|iu) = -i(u|u)$$

and since $(u|u)$ is real it must be equal to zero. Identical deduction for $A - iI$ gives $Ker(A \pm iI) = \{0\}$.

2 ⇒ 3 : Clearly the domains of A and $A \pm iI$ coincide and if $v \in D(A^*)$ then also $v \in D((A \pm iI)^*)$. Thus $A \pm iI$ and $(A \pm iI)^*$ are both densely defined. Moreover, since

$$((A \pm iI)u|v) = (u|(A \pm iI)^* v)$$

we have

$$(Au|v) = (u|((A \pm iI)^* \pm iI)v)$$

for $u \in D(A)$ and $v \in D((A \pm iI)^*)$. Now it follows that $D((A \pm iI)^*) = D(A^*)$ and $A^* = ((A \pm iI)^* \pm iI)$ or equivalently $A^* \mp iI = (A \pm iI)^*$.

Now let v be an element of $\overline{Ran(A - iI)}^\perp$. By lemma 3.13 and the above arguments

$$H = Ker(A^* + iI) \oplus \overline{(Ran(A - iI))}$$

and thus $v \in Ker(A^* + iI) = \{0\}$.

We still need to prove that $Ran(A - iI)$ is closed. Let $u \in D(A)$ and note that

$$\|(A - iI)u\|^2 = \|Au\|^2 + (Au| - iu) + (-iu|Au) + \|u\|^2 = \|Au\|^2 + \|u\|^2 \tag{3.5}$$

by the fact that A was assumed to be symmetric. Let $\{u_k\}_{k=1}^{\infty}$ be a sequence in $D(A)$ such that the elements $(A - iI)u_k$ converge to some $w \in H$. By (3.5) we have

$$\|(A - iI)(u_k - u_j)\|^2 = \|A(u_k - u_j)\|^2 + \|u_k - u_j\|^2$$

and thus by completeness and linearity of A both u_k and Au_k converge to some u and v in H respectively. Since A is assumed to be a closed operator it follows that $v = Au$ and therefore

$$w = \lim_{k \to \infty} (A - iI)u_k = Au - iu.$$

Hence $w \in Ran(A - iI)$. The case for $A + iI$ is identical.

3 $\Rightarrow$ *1* : We aim to show that $D(A^*) \subset D(A)$ which will give *1*. If $u \in D(A^*)$ by *3* there exists an element v of $D(A)$ such that

$$(A^* - iI)u = (A - iI)v \tag{3.6}$$

Since A is symmetric $D(A) \subset D(A^*)$ and thus $(u - v) \in D(A^*)$. Hence $u - v$ is in the kernel of $A^* - iI$ by the equation (3.6). By lemma 3.13

$$H = Ker(A^* - iI) \oplus \overline{Ran(A - iI)} = Ker(A^* - iI) \oplus H$$

and thus $Ker(A^* - iI)$ is orthogonal to H which can only be if $Ker(A^* - iI) = \{0\}$. We have proven $u = v \in D(A)$ and thus A is self-adjoint. □

Definition 3.15. *If A is an operator on H such that $(Au|u) \geq c\|u\|^2$ for some $c > 0$ and all $u \in D(A)$ then A is said to be strictly positive.*

For a strictly positive operator theorem 3.14 has a slightly simpler form.

Theorem 3.16. *Let A be a symmetric and strictly positive operator on a Hilbert space H. Then the following conditions are equivalent.*

1. *A is essentially self-adjoint.*
2. *$Ker(A^*) = \{0\}$.*
3. *The range of A is dense in H.*

Proof. *1* $\Rightarrow$ *2*: As A is strictly positive its closure $\overline{A}$ preserves the lower bound. Thus if $u \in Ker(A^*)$ then

$$c\|u\|^2 \leq (\overline{A}u|u) = (A^*u|u) = 0$$

so that $u = 0$.

2 $\Leftrightarrow$ *3*: Since A is symmetric it is closable and thus the adjoint is densely defined by theorem 3.8. If $Ker(A^*) = 0$, we can use lemma 3.13 to get

$$H = Ker(A^*) \oplus \overline{Ran(A)} = \{0\} \oplus \overline{Ran(A)} = \overline{Ran(A)}.$$

In the other direction we use the same orthogonal decomposition which immediately gives that $Ker(A^*) = 0$.

2 $\Rightarrow$ *1*: As A is symmetric and thus also closable we have that $\overline{A} \subset A^*$. We will show that the range of $\overline{A}$ is closed.

To this end, let $(u_k)_{k=1}^{\infty} \subset Ran(\overline{A})$ be a sequence converging to some $u \in H$ and denote by v_k the elements in $D(\overline{A})$ for which $u_k = \overline{A}v_k$. By the Cauchy-Schwartz inequality and the fact that $\overline{A}$ is strictly positive we get

$$c\|v_k - v_j\|^2 \leq (\overline{A}v_k - \overline{A}v_j|v_k - v_j) \leq \|\overline{A}v_k - \overline{A}v_j\|\|v_k - v_j\|$$

and so

$$c\|v_k - v_j\| \leq \|\overline{A}v_k - \overline{A}v_j\|.$$

Thus $(v_k)_{k=1}^{\infty}$ is a Cauchy sequence and by completeness it converges to some $v \in H$.

From this it follows that $(v_k, \overline{A}v_k)_{k=1}^{\infty}$ is a convergent sequence in $H \times H$ and as the graph of $\overline{A}$ is by definition closed we must have that $(v, u) \in \mathcal{G}(\overline{A})$. Now $u = \overline{A}v \in Ran(\overline{A})$ and thus the range of $\overline{A}$ is a closed set in H.

We again utilize lemma 3.13 to get

$$H = Ran(\overline{A}).$$

From this and the fact that $\overline{A}$ is symmetric we have for each $u \in D(A^*)$ an element $u' \in D(\overline{A})$ such that

$$A^*u' = \overline{A}u' = A^*u.$$

Since A^* is injective u and u' must be equal. This proves that $D(A^*) \subset D(\overline{A})$ and so $\overline{A} = A^*$. $\square$

Theorem 3.17. *Let A be a densely defined symmetric operator such that its inverse exists and is also densely defined. Then the adjoint of A is also invertible and the self-adjointness of A is equivalent to that of A^{-1}.*

Proof. If $u \in D(A)$ and $v \in D((A^{-1})^*)$ we have

$$(u, v) = (A^{-1}Au, v) = (Au, (A^{-1})^* v)$$

and so $(A^{-1})^* v \in D(A^*)$ with $A^*(A^{-1})^* v = v$.

Similarly if $u \in D(A^{-1})$ and $v \in D(A^*)$ we get

$$(u, v) = (AA^{-1}u, v) = (A^{-1}u, A^* v).$$

Thus $A^* v \in D((A^{-1})^*)$ with $(A^{-1})^* A^* v = v$. From these two observations it follows that the inverse of A^* exists and that $(A^*)^{-1} = (A^{-1})^*$. From this it is quite easy to see that the self-adjointness of A is equivalent to that of A^{-1}. □

3.4 The spectrum of an operator

We will now define the spectrum of an operator which generalizes the concept of an eigenvalue from finite dimensional linear algebra.

Definition 3.18. *Let A be a densely defined operator on a Hilbert space H. Then*

$$\varrho(A) = \{z \in \mathbb{C} \ : \ \text{the operator } (A - zI)^{-1} : H \to D(A) \text{ exists and is bounded in } H\}$$

is called the resolvent set of A. For $z \in \varrho(A)$ we also denote $R_z = (A - zI)^{-1}$ and call this the resolvent of A. The spectrum $\sigma(A)$ is now defined as the set $\mathbb{C} \setminus \varrho(A)$.

We will show that the spectrum of a self-adjoint operator is real. This will be used in the proof of the Kato-Rellich theorem.

Theorem 3.19. *Let A be a self-adjoint operator. Then $z \in \varrho(A)$ if and only if $A - zI$ is bounded from below. That is, there exists a constant $C > 0$ such that*

$$\|(A - zI)u\| \geq C\|u\|$$

for all $u \in H$. Moreover, the spectrum of a self-adjoint operator is a subset of $\mathbb{R}$.

Proof. Fix $z \in \varrho(A)$. By definition there exists a constant $c > 0$ such that

$$\|R_z v\| \leq c\|v\|$$

for all $v \in H$. Now $R_z(A - zI)u = u$ for all $u \in D(A)$ and thus

$$\|u\| = \|R_z(A - zI)u\| \leq c\|(A - zI)u\|.$$

Therefore the "only if" part of the claim holds with the constant c^{-1}.

Now assume that $(A - zI)$ is bounded from below by some constant $C > 0$. Note that it is thus clearly an injection.

We will prove that $Ran(A - zI) = H$. Assume that this is not the case, i.e., there is some non-zero element $v \in H$ such that $v \perp Ran(A - zI)$. Now

$$0 = (v|(A - zI)u) = (v|Au) - (v|zu)$$

for all $u \in D(A)$ and hence

$$(v|Au) = (v|zu) = (\overline{z}v|u)$$

which implies that $v \in D(A^*)$ and

$$A^*v = Av = \overline{z}v.$$

Since

$$\|(A - zI)v\|^2 = \|Av\|^2 - z(v|Av) - \overline{z}(v|Av) + |z|^2\|v\|^2 \tag{3.7}$$

$$= \|Av\|^2 + |z|^2\|u\|^2 - 2\Re(z)(v|Av),$$

we have the equality

$$\|(A - zI)v\| = \|(A - \overline{z}I)v\|.$$

As A is self-adjoint,

$$0 = \|(A - \overline{z}I)v\| \geq C\|v\|$$

and thus $v = 0$. So $A - zI$ is a bijection.

Now let $u \in H$ and note that

$$\|u\| = \|(A - zI)(A - zI)^{-1}u\| \geq C\|(A - zI)^{-1}u\|.$$

Thus $(A - zI)^{-1}$ is also bounded. This proves the "if" part.

Lastly, by a similar calculation as (3.7) we have

$$\|(A - zI)u\|^2 = \|(A - \Re(z)I)u - i\Im(z)u\|^2 = \|(A - \Re(z)I)u\|^2 + \|i\Im(z)u\|^2 \geq |\Im(z)|^2\|u\|^2$$

for all $u \in D(A)$. As we just showed, this implies $z \in \varrho(A)$ whenever $\Im(z) \neq 0$.

$\square$

Chapter 4

Distributions

When dealing with differential operators one finds that usually the classical derivative is too restrictive to make the operators self-adjoint. The domain of the adjoint often turns out to be a space of functions for which a weaker form of derivative exists and has some integrability properties. Here we briefly introduce distributions and some basic results. As the construction of the test function space and especially its topology is quite complicated and would take much of the length of this thesis, we will not go in to the subject in detail and for most proofs the reader will be referred to a source.

A thorough treatment of distributions can be found, for example, in chapter 6 of [5] which will also be our main source. Similar construction is also given by Simon and Reed in [2].

Throughout this chapter Ω will denote a non-empty open subset of $\mathbb{R}^d$ and $\mathbb{K}$ denotes either of the fields $\mathbb{R}$ or $\mathbb{C}$.

4.1 Topological vector spaces

Definition 4.1. *Let X be a vector space endowed with a topology τ that satisfies the following conditions:*

1. *Every set with only one element is closed in the topology τ.*

2. *The vector space operations are continuous with respect to τ in the sense of product topologies on $X \times X$ and $\mathbb{K} \times X$.*

Then X is said to be a topological vector space or abbreviated TVS.

This is the definition given by Rudin in [5]. Some authors do not include the first condition in the definition of a topological vector space but all examples we are going to encounter will satisfy it. In particular the first condition ensures that all topological vector spaces are Hausdorff spaces.

A base for a topology τ is a collection $\mathcal{B}$ of open sets such that every set in τ can be expressed as a union of sets in $\mathcal{B}$. A local base of an element x in a topological space is a collection $\mathcal{B}_x$ of open neighbourhoods of x such that every open set containing x contains an element of $\mathcal{B}_x$ as a subset.

For a topological vector space X a local base $\mathcal{B}_x$ actually defines a base for its entire topology. This is due to the fact that in such a space the topology is translation invariant with respect to the vector addition and scalar multiplication operations. Meaning that, as maps from X to itself

$$T_a : x \longmapsto x + a$$

and

$$M_\alpha : x \longmapsto \alpha x$$

preserve open sets for all $a \in X$ and $\alpha \in \mathbb{K} \setminus \{0\}$. It is not hard to see that they are both actually homomorphisms. Thus, if V is a neighbourhood of the zero vector, then $x + V$ is a neighbourhood of the vector $x \in X$. Therefore a base is given by the collection of all sets of the type $\bigcup_{\alpha \in I, \beta \in J}(x_\alpha + U_\beta)$ where I and J are some index sets, $x_\alpha \in X$ for all $\alpha \in I$ and $U_\beta \in \mathcal{B}$ for all $\beta \in J$.

From now on, a local base is understood to mean a local base at zero and whenever a local base for a topological vector space is given, we assume that the topology is generated in the above fashion.

Theorem 4.2. *A linear map $\Lambda : X \to Y$ between two topological vector spaces X and Y is continuous if and only if it is continuous at zero.*

Proof. Let $x \in X$ and let $V \subset Y$ be a neighbourhood of the vector Λx. Then $-\Lambda x + V$ is a neighbourhood of the zero vector in Y and there exists a neighbourhood of zero W in X such that $\Lambda W \subset -\Lambda x + V$. For any $w \in W$ we have

$$\Lambda(x + w) = \Lambda x + \Lambda w = \Lambda x - \Lambda x + v = v$$

for some $v \in V$. Thus $x + W$ is a neighbourhood of x that gets mapped in to V by Λ and therefore Λ is continuous at x. □

Definition 4.3. *Let U be a subset of a vector space X.*

1. *If $\alpha U := \{\alpha u | u \in U\} \subset U$ for every $\alpha \in \mathbb{K}$ with $|\alpha| \leq 1$, then U is said to be balanced.*

2. *If $(1 - \alpha)u + \alpha v \in U$ for all $u, v \in U$ and $0 \leq \alpha \leq 1$, then U is said to be convex.*

3. *If there exists a scalar $\beta \in \mathbb{K}$ for every neighbourhood V of the zero vector such that $U \subset \beta V$, then U is said to be bounded.*
4. *If every $x \in X$ is an element of the set rU for some positive real number r then, U is said to be absorbing.*

By the definition of a topological vector space, the multiplication operation is continuous. Thus, for a fixed $x \in X$ the map $f_x : \mathbb{K} \to X$ where $f_x(\alpha) = \alpha x$ is continuous. If $V \subset X$ is a neighbourhood of the zero vector then $f_x^{-1}(V) = \{\alpha \in K : \alpha x \in V\}$ is an open set in $\mathbb{K}$ containing zero. Recall that $\mathbb{K}$ denotes either of the fields $\mathbb{R}$ or $\mathbb{C}$. Therefore there is a natural number N such that $\frac{1}{n} \in f_x^{-1}(V)$ for all $n > N$ and so $x \in nV$ for all $n > N$. Thus every neighbourhood of the zero vector is absorbing in a topological vector space. From this it follows that every converging sequence in a topological vector space is a bounded set.

Definition 4.4. *A topological vector space (X, τ) is said to be locally convex if τ has a local base whose elements are convex.*

Theorem 4.5. *Every topological vector space has a balanced local base and every locally convex topological space has a convex and balanced local base.*

Proof. For the first claim, let X be a topological vector space and let $U \subset X$ be a neighbourhood of the zero vector. Since the multiplication operation $M(\alpha, x) = \alpha x$ is a continuous map from $\mathbb{K} \times X$ to X, there is an open set $W \subset \mathbb{K} \times X$ such that $M(W) \subset U$ and $0 \in M(W)$. In particular $B(0, r) \times V \subset W$ for some $r > 0$ and open set $V \subset X$. Now

$$B = \bigcup_{|\alpha|<r} \alpha V$$

is a balanced neighbourhood of zero and $B \subset U$.

Thus every neighbourhood of the zero vector contains a balanced neighbourhood of zero and thus X has a balanced local base.

Now let X be a locally convex topological vector space and let $U \subset X$ be a convex open set containing zero. Let

$$A = \bigcap_{\alpha \in \partial B(0,1)} \alpha U$$

where $\partial B(0, 1)$ is the boundary of the unit ball in $\mathbb{K}$. It is not hard to show that scalar multiples and intersections of convex sets remain convex. Thus A is convex.

Let B be open and balanced subset of U, the existence of which we proved earlier. Now for $\alpha \in \partial B(0, 1)$ we have $|\alpha^{-1}| = |\overline{\alpha^{-1}}| = 1$ and so

$$x = \alpha^{-1}\overline{\alpha^{-1}}x$$

for all $x \in X$ and since B is balanced we have $\alpha^{-1}B = B$. Since $B \subset U$ we get $B \subset \alpha U$ for all $\alpha \in \partial B(0,1)$ and thus $B \subset A$. Now A^o is a non-empty open set containing zero and $A^o \subset U$. If $0 < r < 1$ we have

$$(1-r)A^o + (r)A^o \subset A$$

and both of the sets $(1-r)A^o$ and rA^o are open since the multiplication map M_α is a homomorphism for non-zero α. Since the sum of any set and an open set is open in a topological vector space[1] we have $(1-r)A^o + rA^o \subset A^o$. Therefore A^o is a convex set.

Let $\gamma \in \mathbb{K}$ such that $|\gamma| \leq 1$ We can write $\gamma = r\beta$ where $r = |\gamma|$ and $\beta \in \mathbb{K}$ with $|\beta| = 1$. Thus $\beta \partial B(0,1) = \partial B(0,1)$ and

$$\gamma A = \bigcap_{\alpha \in \partial B(0,1)} r\beta\alpha U = \bigcap_{\alpha \in \partial B(0,1)} r\alpha U.$$

Since αU is convex and contains the zero element for all α, we have

$$(1-r)x + ry \in \alpha U$$

for all $x, y \in U$. In particular $ry \in \alpha U$ for all $y \in \alpha U$. Thus $r\alpha U \subset \alpha U$ for all $\alpha \in \partial B(0,1)$ and we can conclude that A is a balanced set which implies that A^o is also balanced. Therefore A^o is a convex and balanced neighbourhood of the zero vector and $A^o \subset U$. $\square$

We can define Cauchy sequences without appealing to a metric or a norm in topological vector spaces.

Definition 4.6. *Let X be a topological vector space and let $\mathcal{B}$ be a local base at the origin. Then a sequence $(x_n)_{n=1}^\infty \subset X$ is said to be Cauchy if for every $B \subset \mathcal{B}$ there is a natural number N such that $(x_n - x_m) \in B$ for all $n, m \in \mathbb{N}$ whenever $n, m \geq N$.*

We note that the above definition coincides with the usual one involving metrics whenever the topology on X is metrizable.

In chapter 1 of [5] it is proved that a topological vector space (X, τ) is metrizable if and only if it has a countable base, and when X is metrizable there always exists a translation invariant metric compatible with τ.

Theorem 4.7. *Let X be a metrizable topological vector space. If $(x_k)_{k=1}^\infty$ is a sequence in X that converges to 0, then there exists an unbounded increasing sequence $(\alpha_k)_{k=1}^\infty$ of scalars such that $\alpha_k > 0$ for all $k \in \mathbb{N}$ and $\alpha_k x_k \to 0$ in X.*

[1] If A is a set and B is an open set in a TVS X then $A + B = \bigcup_{a \in A}(a + B)$.

Proof. Denote by d the translation invariant metric compatible with the topology of X. Now by iterating the triangle inequality and using translation invariance we get

$$d(nx,0) \leq d(nx,(n-1)x) + d((n-1)x,0) \leq ...$$
$$\leq \sum_{k=1}^{n} d(kx,(k-1)x) = \sum_{k=1}^{n} d(x,0) = nd(x,0)$$

for all $x \in X$.

Let $(x_k)_{k=1}^{\infty}$ converge to 0 in X. We may also assume that for every $n \in \mathbb{N}$ there is an index $k > n$ such that $x_k \neq 0$ since otherwise any unbounded scalar sequence will do.

Now for every neighbourhood of zero V there exists a natural number M such that $x_k \in V$ for all $k > M$. Since d is compatible with the topology of X, the open balls centered at the origin form a local base and we have $d(x_k, 0) \to 0$ as $k \to \infty$. Thus for every $N \in \mathbb{N}$ there exists a k_N such that $d(x_k, 0) < N^{-2}$ whenever $k > k_N$. Notice that the numbers k_N form an increasing sequence since the original sequence contains infinitely many non-zero elements.

Now we define the scalars α_k as follows: If $k < k_1$ then $\alpha_k = 1$, and if $k_N \leq k < k_{N+1}$ then $\alpha_k = N$. By the use of the inequality from the start of the proof, for any $N \in \mathbb{N}$ we get

$$d(\alpha_k x_k, 0) \leq \alpha_k d(x_k, 0) \leq \alpha_k N^{-2} \leq N^{-1}$$

whenever $k > k_N$. Thus $\alpha_k x_k \to 0$ in X.

□

4.2 Test functions and distributions

Definition 4.8. *Let $\Omega \subset \mathbb{R}^d$ be a non-empty open set and K a compact subset of Ω. We define $\mathcal{D}_K(\Omega)$ to be the space of functions in $C^\infty(\Omega)$ whose support lies in K.*

Let $\{K_n\}_{0=1}^{\infty}$ be a sequence of compact subsets of Ω with non-empty interiors such that $\mathbf{int}K_n \subset \mathbf{int}K_{n+1}$ and $\cup_{n=0}^{\infty} K_n = \Omega$. A collection of such sets always exists for any open subset of $\mathbb{R}^d$. For example, with large enough $n \in \mathbb{N}_0$ the sets

$$K_n = \overline{B(0,n+1)} \cap \{x \in \mathbb{R}^d : d(x,\Omega^c) \geq (n+1)^{-1}\}$$

form such a sequence.

Now for every $n \in \mathbb{N}_0$ define

$$p_n(f) = \max_{x \in K_n, |\alpha| \leq n} |D^\alpha f(x)|. \tag{4.1}$$

The p_n above are seminorms on the space $C^\infty(\Omega)$ since they have all the properties of a norm except that there can exist non-zero functions $f \in C^\infty(\Omega)$ for which $p_n(f) = 0$. For example,

$$\phi(x) = \begin{cases} e^{-(x^2-1)^{-1}} & |x| < 1 \\ 0 & |x| \geq 1 \end{cases}$$

is in $C^\infty(\mathbb{R})$ and the sets $K_n = [-2-n, -1+n]$ satisfy the properties stated above but $p_0(\phi) = 0$.

Notice that if $p_n(f) = 0$ for very $n \in \mathbb{N}_0$ then necessarily $f = 0$. A collection of seminorms with this property is said to be separating.

The following theorem gives us a way to define a topology to a space when a separating collection of seminorms is given.

Theorem 4.9. *Let X be a vector space with a countable separating collection $\mathcal{P}$ of seminorms and define*

$$V(p, N) = \{x \in X : p(x) < N^{-1}\}$$

for all $p \in \mathcal{P}$ and $N \in \mathbb{N}$. Then the collection of finite intersections of the sets $V(p, N)$ forms a convex and balanced local base for a translation invariant metrizable topology τ on X. The space (X, τ) is a locally convex topological vector space and additionally,

1. *every seminorm $p \in \mathcal{P}$ is a continuous functional on this space,*

2. *a set $V \subset X$ is bounded if and only if every seminorm in $\mathcal{P}$ is bounded on V.*

Proof. See theorems 1.24 and 1.37 of [5]. □

In the case where the seminorms are increasing in n a local base is generated by the sub-collection consisting of the sets

$$V(N) := V(p_N, N) = \{x \in X : p_N(x) < \frac{1}{N}\}.$$

By theorem 4.9 we now have a metrizable topology for $C^\infty(\Omega)$. It turns out that with this metric $C^\infty(\Omega)$ becomes a complete metric space with the Heine-Borel property. The subspaces $\mathcal{D}_K(\Omega)$ of $C^\infty(\Omega)$ are closed in this topology since the functionals

$$\Lambda_x : \phi \longmapsto \phi(x)$$

are continuous for all $x \in \Omega$ and

$$\mathcal{D}_K(\Omega) = \bigcap_{x \in K^c} Ker(\Lambda_x).$$

Thus as a subspace $\mathcal{D}_K(\Omega)$ is also complete for all compact $K \subset \Omega$. For details see chapter 1 of [5]. We will denote the subspace topologies of $\mathcal{D}_K(\Omega)$ by τ_K from now on.

A space, whose topology is defined by a collection of seminorms that is also complete is called a Fréchet space.

Next we define the space of test functions.

Definition 4.10. *Denote*

$$\mathcal{D}(\Omega) = \bigcup_K \mathcal{D}_K(\Omega)$$

where K goes through all the compact subsets of Ω. Elements of the resulting vector space $\mathcal{D}(\Omega)$ are called test functions.

We can define a sequence of norms for $\mathcal{D}(\Omega)$ with

$$\|\phi\|_n = \sup_{x\in\Omega, |\alpha|\leq n} |D^\alpha \phi(x)|$$

where $n \in \mathbb{N}_0$. When restricted to any of the subspaces $\mathcal{D}_K(\Omega)$ they induce the same topology as the seminorms in (4.1). This can be seen from the fact that if $\phi \in \mathcal{D}_K(\Omega)$ is fixed, the support of ϕ lies in some compact $K \subset \Omega$ and there exists some $n_0 \in \mathbb{N}_0$ for which $K \subset K_n$ for all $n \geq n_0$. Thus for all such n we have $\|\phi\|_n = p_n(\phi)$. Since both $\|\phi\|_n$ and $p_n(\phi)$ are increasing in n we have

$$V(N+1) \subset V(N)$$

for both of the local bases generated by $\|\cdot\|_n$ and p_n respectively. Thus the resulting topology does not change if we start the semi-norm sequences from n_0.

Again by theorem 4.9 the collection of norms $\|\cdot\|_n$ assigns a metrizable topology to the space $\mathcal{D}(\Omega)$ but sadly this metric will not be complete. Let $\eta_0 \in C_c^\infty(\mathbb{R})$ be the function from example 6.2. Then

$$\phi_k(x) = \sum_{n=1}^{k} n^{-1}\eta_0(x-2n)$$

is an element of $\mathcal{D}(\mathbb{R})$. Fix $N \in \mathbb{N}$ and let α be a multi-index such that $|\alpha| \leq N$. Now for $j, k \in \mathbb{N}$ with the arbitrary choice that $k \geq j$ we have

$$\Big|D^\alpha(\phi_k(x) - \phi_j(x))\Big| = \Big|\sum_{n=j+1}^{k} n^{-1} D^\alpha \eta_0(x-2n)\Big| \leq \frac{1}{(j+1)}\Big|\sum_{n=j+1}^{k} D^\alpha \eta_0(x-2n)\Big|$$

where only one term in the last expression is non-zero at any given $x \in \mathbb{R}$. Thus

$$\Big|D^\alpha(\phi_k(x) - \phi_j(x))\Big| \leq \frac{1}{(j+1)} \max_{|\alpha|\leq N} \|D^\alpha \eta_0\|_\infty$$

where the maximum is a constant when N is fixed since η_0 is compactly supported. From this we see that $\phi_k - \phi_j \in V(N)$ whenever $k, j > C_N N$ where $C_N = \max_{|\alpha| \leq N} \|D^\alpha \eta_0\|_\infty$. Thus the functions ϕ_k form a Cauchy sequence in $\mathcal{D}(\mathbb{R})$ but their limit cannot be compactly supported since $supp(\phi_k) \subset supp(\phi_{k+1})$ and $\phi_{k+1}|_{supp(\phi_k)} = \phi_k$ for all $k \in \mathbb{N}$ and $\bigcup_{k=1}^\infty supp(\phi_k) = (-\infty, -1]$.

However another non-metrizable topology τ can be constructed with the use of strict inductive limits on locally convex spaces. This topology turns out to have the nice property that all Cauchy sequences converge and also the restrictions of τ to the subspaces $\mathcal{D}_K(\Omega)$ will give the same topology that we constructed earlier through the use of the seminorms p_n.

Definition 4.11. *Denote by $\mathcal{B}$ the collection of convex balanced sets W in $\mathcal{D}(\Omega)$ such that $\mathcal{D}_K(\Omega) \cap W \in \tau_K$ for all compact sets $K \subset \Omega$. Then we denote by τ the collection of all unions of sets that are of the form $\phi + W$ where $\phi \in \mathcal{D}(\Omega)$ and $W \in \mathcal{B}$.*

Theorem 4.12.

1. *The collection τ is a topology on $\mathcal{D}(\Omega)$ and $\mathcal{B}$ forms a local base for it.*

2. *The pair $(\mathcal{D}(\Omega), \tau)$ is a locally convex topological vector space.*

Proof. See theorem 6.4 in [5]

□

From now on we will always assume that $\mathcal{D}(\Omega)$ is equipped with the topology τ described above. This topology might seem cumbersome to work with but the next theorem tells us that in most cases we can restrict our study to some suitable subspace $\mathcal{D}_K(\Omega)$ where the topology τ_K can be used.

Theorem 4.13.

1. *If $V \subset \mathcal{D}(\Omega)$ is convex and balanced then it is open if and only if $V \in \mathcal{B}$.*

2. *The subspace topology of $\mathcal{D}_K(\Omega) \subset \mathcal{D}(\Omega)$ induced by τ equals τ_K.*

3. *For every bounded set $U \in \mathcal{D}(\Omega)$ there exists a compact set $K \subset \Omega$ such that $U \subset \mathcal{D}_K(\Omega)$ and for all $n \in \mathbb{N}_0$ there exists $M_n < \infty$ such that*

$$\|\phi\|_n \leq M_n$$

for all $\phi \in U$.

4. *Every closed and bounded subset of $\mathcal{D}(\Omega)$ is compact. That is, $\mathcal{D}(\Omega)$ has the Heine-Borel property.*

5. *A Cauchy sequence $(\phi_k)_{k=1}^{\infty}$ in $\mathcal{D}(\Omega)$ is always contained in some $\mathcal{D}_K(\Omega)$ and is also Cauchy with respect to the seminorms $\|\cdot\|_n$ for all $n \in \mathbb{N}_0$.*

6. *If $\lim_{k\to\infty} \phi_k = 0$ in $\mathcal{D}(\Omega)$ then the support of every ϕ_k is contained in some compact set $K \subset \Omega$ and $D^\alpha \phi_k$ converges to zero uniformly for every multi-index α as $k \to \infty$.*

7. *Every Cauchy sequence in $\mathcal{D}(\Omega)$ converges.*

Proof. See theorem 6.5 of [5]. □

Definition 4.14. *The space of distributions, denoted by $\mathcal{D}'(\Omega)$, is defined as the set of continuous linear functionals on $\mathcal{D}(\Omega)$.*

Theorem 4.15. *Let Y be a locally convex topological vector space and let $\Lambda : \mathcal{D}(\Omega) \to Y$ be linear. Then the following properties are equivalent:*

1. *Λ is continuous.*

2. *If $U \subset \mathcal{D}(\Omega)$ is bounded, then the image of U under Λ is bounded in Y.*

3. *If a sequence $(\phi_k)_{k=1}^{\infty} \subset \mathcal{D}(\Omega)$ converges to zero then $\Lambda\phi_k \to 0$ in Y as $k \to \infty$.*

4. *$\Lambda|_{\mathcal{D}_K(\Omega)}$ is continuous in $\mathcal{D}_K(\Omega)$ for all compact $K \subset \Omega$.*

Proof.

$1 \Rightarrow 2$: Let $U \in \mathcal{D}(\Omega)$ be bounded and let $V \subset Y$ be a neighbourhood of the zero vector. Since Λ is continuous the set $W := \Lambda^{-1}V \subset \mathcal{D}(\Omega)$ is a neighbourhood of zero and thus there exists a $\alpha \in \mathbb{C}$ such that $U \subset \alpha W$. Therefore

$$\Lambda U \subset \Lambda \alpha W = \alpha \Lambda W = \alpha V$$

i.e., ΛU is a bounded set.

$2 \Rightarrow 3$: Let $(\phi_k)_{k=1}^{\infty}$ be sequence in $\mathcal{D}(\Omega)$ converging to zero. By theorem 4.13 there is some compact set $K \subset \Omega$ such that the support of every ϕ_k is contained in it. When Λ is restricted to $\mathcal{D}_K(\Omega)$ the boundedness property remains since every bounded set in $\mathcal{D}(\Omega)$ is contained in some subspace $\mathcal{D}_K(\Omega)$. Recall that the topology τ_K was metrizable and translation invariant. Thus we can use theorem 4.7 to obtain an unbounded increasing sequence of positive real numbers r_k such that $r_k\phi_k \to 0$ in $\mathcal{D}_K(\Omega)$. Since $(r_k\phi_k)_{k=1}^{\infty}$ forms a bounded set then so does $B = \{\Lambda r_k \phi_k : k \in \mathbb{N}\}$.

Let $V \subset Y$ be a balanced neighbourhood of 0. Then there exists some non-zero $\gamma \in \mathbb{C}$ such that $B \subset \gamma V$ and thus $\gamma^{-1}B \subset V$. Now there exists some $N \in \mathbb{N}$ such that $r_k^{-1}\gamma < 1$ for all $k > N$. Thus

$$\gamma^{-1}\Lambda r_k \phi_k \in \gamma^{-1}B \subset V$$

for all $k \in \mathbb{N}$ and so

$$r_k^{-1}\Lambda r_k \phi_k \in V$$

when $k > N$ as V is a balanced set. We have shown that $r_k^{-1}\Lambda r_k \phi_k = \Lambda\phi_k$ converges to zero in Y.

$3 \Rightarrow 4$: The spaces $\mathcal{D}_K(\Omega)$ are metrizable for every compact $K \subset \Omega$ and thus sequential continuity is equivalent to continuity in their respective topologies. Let $(\phi_k)_{k=1}^{\infty}$ be a sequence converging to 0 in $\mathcal{D}_K(\Omega)$ for some compact $K \subset \Omega$. Since the topology τ_K coincides with τ restricted to the subspace $\mathcal{D}_K(\Omega)$ by theorem 4.13, we have convergence also in $D(\Omega)$. Thus $\Lambda\phi_k \to 0$ in Y by assumption and therefore Λ is continuous when restricted to $\mathcal{D}_K(\Omega)$.

$4 \Rightarrow 1$: Let $V \subset Y$ be a convex and balanced neighbourhood of the zero vector. Then $W = \Lambda^{-1}V$ is also convex and balanced, and $\mathcal{D}_K(\Omega) \cap W \in \tau_K$ for all compact $K \subset \Omega$ since $\Lambda|_{\mathcal{D}_K(\Omega)}$ is assumed to be continuous. By the first property of theorem 4.13 this implies that W is an open set in $\mathcal{D}(\Omega)$. By theorem 4.5 the space Y has a convex and balanced local base and thus Λ is continuous since we just proved that the inverse images of the base elements are open in $\mathcal{D}(\Omega)$.

□

An immediate consequence of theorem 4.15 is that all differential operators

$$D^\alpha : \mathcal{D}(\Omega) \to \mathcal{D}(\Omega)$$

are continuous: Let $\phi \in \mathcal{D}(\Omega)$ then,

$$\|D^\alpha \phi\|_N = \sup_{x\in\Omega,|\beta|\le N} |D^{\beta+\alpha}\phi(x)| \le \sup_{x\in\Omega,|\beta|\le N+|\alpha|} |D^\beta \phi(x)| = \|\phi\|_{N+|\alpha|}$$

for all $N \in \mathbb{N}$. For a sequence $(\phi_k)_{k=1}^{\infty}$ converging to 0 in some subspace $\mathcal{D}_K(\Omega)$ we have for every $V(N)$ a natural number M such that $\phi_k \in V(N + |\alpha|)$ for all $k > M$. Then $D^\alpha\phi_k \in V(N)$ for all $k > M$. Thus D^α is continuous in every subspace $\mathcal{D}_K(\Omega)$.

Theorem 4.16. *Let $\Lambda : \mathcal{D}(\Omega) \to \mathbb{C}$ be linear. Then Λ is a distribution if and only if for every compact $K \subset \Omega$ there exists a natural number N and a positive constant C such that*

$$|\Lambda\phi| \leq C\|\phi\|_N$$

for all $\phi \in \mathcal{D}_K(\Omega)$.

Proof. Assume first that $K \subset \Omega$ is compact and there exists $N \in \mathbb{N}$ and $C > 0$ such that the described inequality holds in $\mathcal{D}_K(\Omega)$. Let $\phi_k \to 0$ in $\mathcal{D}_K(\Omega)$ and fix $n \in \mathbb{N}$ with $n > N$ Now there exists some $M_n \in \mathbb{N}$ such that $\phi_k \in V(n)$ for all $k > M_n$. We have

$$|\Lambda\phi_k| \leq C\|\phi_k\|_N \leq C\|\phi_k\|_n \leq \frac{C}{n}$$

whenever $k > M_n$. So Λ is a distribution by theorem 4.15.

For the other direction assume that Λ is a distribution and that for some compact $K \subset \Omega$ the described constants do not exist. I.e., there is a sequence of non-zero elements $\phi_k \in \mathcal{D}_K(\Omega)$ such that $|\Lambda\phi_k| \geq k\|\phi_k\|_k$ for all k. Therefore

$$\Big|\Lambda(k^{-1}\frac{\phi_k}{\|\phi_k\|_k})\Big| \geq 1,$$

but for $N < k$ we have

$$\Big\|k^{-1}\frac{\phi_k}{\|\phi_k\|_k}\Big\|_N = k^{-1}\frac{\|\phi_k\|_N}{\|\phi_k\|_k} \leq k^{-1} < N^{-1}$$

and thus $(k\|\phi_k\|_k)^{-1}\phi_k$ converges to 0 in $\mathcal{D}_K(\Omega)$. This is a contradiction since Λ was assumed to be continuous in $\mathcal{D}(\Omega)$ and therefore also in $\mathcal{D}_K(\Omega)$ by theorem 4.15. $\square$

To conclude this chapter we list a few examples.

Example 4.17.

- When $x \in \Omega$, the map $\delta_x : \mathcal{D}(\Omega) \to \mathbb{C}$ defined as

  $$\delta_x\phi = \phi(x),$$

  is a distribution by theorem 4.16 since

  $$|\delta_x\phi| = |\phi(x)| \leq \|\phi\|_N$$

  for all $N \in \mathbb{N}_0$. The distribution δ_x is called the Dirac measure at the point x.

- Let $f \in L^p_{loc}(\Omega)$ with $1 \leq p < \infty$. Then $f \in L^1_{loc}(\Omega)$ since

$$\int_K |f|dx = \int_{K_1} |f|dx + \int_{K^1} |f|dx \leq m(K) + \int_K |f|^p dx < \infty$$

where m is the Lebesgue measure and K is any compact subset of Ω with

$$K_1 = \{x \in K : |f(x)| < 1\}$$

and

$$K^1 = \{x \in K : |f(x)| \geq 1\}.$$

Therefore f defines a distribution Λ_f with

$$\Lambda_f \phi = \int_\Omega f\phi dx,$$

since $\phi \in \mathcal{D}_K(\Omega)$ for some compact $K \subset \Omega$ implies

$$|\Lambda_f \phi| = \Big| \int_\Omega f\phi dx \Big| \leq \|\phi\|_0 \int_K |f|dx.$$

When a distribution is defined by a function in the above manner, we identify Λ_f with f. That is, we say f is a distribution instead of distribution induced by f.

4.3 Differentiation of distributions

Definition 4.18. *Let Ω be a non-empty open subset of $\mathbb{R}^d$, Λ a distribution on Ω and $\alpha \in \mathbb{N}^d$. Then we define the operator D^α in $\mathcal{D}'(\Omega)$ by*

$$D^\alpha \Lambda \phi = (-1)^{|\alpha|} \Lambda(D^\alpha \phi)$$

for all $\phi \in \mathcal{D}(\Omega)$. The map $D^\alpha \Lambda$ is called the α:th distributional derivative of Λ.

As one might hope it turns out that every distribution is differentiable in the sense of the derivative defined above. That is, the operator D^α maps $\mathcal{D}'(\Omega)$ to itself for all multi-indices $\alpha \in \mathbb{N}^d$. To see this let $K \subset \Omega$ be compact. Then there exists some $N \in \mathbb{N}$ and $C > 0$ such that

$$|\Lambda\phi| < C\|\phi\|_N$$

for all $\phi \in \mathcal{D}_K(\Omega)$. Thus

$$|D^\alpha \Lambda \phi| = |\Lambda D^\alpha \phi| < C\|D^\alpha \phi\|_N \leq C\|\phi\|_{N+|\alpha|}$$

for all $\phi \in \mathcal{D}_K(\Omega)$ and theorem 4.16 applies.

Let $f \in C^N(\Omega)$ for some $N \in \mathbb{N}$ and α a multi-index such that $|\alpha| \leq N$. Then $D^\alpha f$ is continuous and thus locally integrable, f and $D^\alpha f$ both define distributions Λ_f and $\Lambda_{D^\alpha f}$ and integration by parts gives

$$D^\alpha \Lambda_f = \Lambda_{D^\alpha f}.$$

However, this does not hold in general when $D^\alpha f$ exists but fails to be continuous. See example 6.14 of [5].

4.4 Convolution and distributions

Recall that the convolution of two Lebesgue measurable functions f and g from $\mathbb{R}^d$ to $\mathbb{C}$ is defined as

$$(f * g)(x) = \int_{\mathbb{R}^d} f(y)g(x-y)dy \tag{4.2}$$

whenever the integral is well defined for almost every $x \in \mathbb{R}^d$.

We will now define convolutions of distributions. For this we first fix some notation. Let X be a vector space, Y a set and $h : X \to Y$ a function. We define the operators T_y and R with

$$T_y h(x) = h(x-y) \text{ and } Rh(x) = h(-x)$$

for all $x, y \in X$. Now the convolution of f and g at a point x can be expressed in terms of T_x and R by

$$(f * g)(x) = \int_{\mathbb{R}^d} f(y) T_x R g(y) dy.$$

This motivates the following definition.

Definition 4.19. *Let $\Lambda \in \mathcal{D}'(\mathbb{R}^d)$, $\phi \in \mathcal{D}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$. Then the convolution of Λ and ϕ at the point x is defined as*

$$(\Lambda * \phi)(x) := \Lambda(T_x R\phi).$$

When f is a locally integrable function, the above definition coincides with the usual definition for convolution since

$$(\Lambda_f * \phi)(x) = \Lambda_f(T_x R\phi) = \int_{\mathbb{R}^d} f(y) T_x R\phi(y) dx = \int_{\mathbb{R}^d} f(y)\phi(x-y)dx = (f * \phi)(x).$$

Here the convolution exists and is finite for all $x \in \mathbb{R}^d$ since ϕ has compact support.

We also define the operator T_y pointwise for $\Lambda \in \mathcal{D}'(\mathbb{R}^d)$ and $\phi \in \mathcal{D}'(\mathbb{R}^d)$ by

$$(T_y \Lambda)\phi = \Lambda(T_{-y}\phi).$$

The map $T_y\Lambda$ is clearly linear. Let $K \in \mathbb{R}^d$ be compact and let $\phi \in \mathcal{D}_K(\mathbb{R}^d)$. Then $K - y$ is also a compact set in $\mathbb{R}^d$ and $T_y\phi \in \mathcal{D}_{K-y}(\mathbb{R}^d)$. Let C and N be the related constants given by theorem 4.16 for which the inequality

$$|\Lambda(T_{-y}\phi)| < C\|T_{-y}\phi\|_N$$

holds. Since $D^\alpha(T_{-y}\phi(x)) = D^\alpha(\phi(x+y)) = (D^\alpha\phi)(x+y)$ for every suitable multi-index α and $x \in \mathbb{R}^d$, we have that

$$\|T_{-y}\phi\|_N = \|\phi\|_N.$$

Thus

$$|(T_y\Lambda)\phi| = |\Lambda(T_{-y}\phi)| < C\|\phi\|_N$$

and so the map $T_y\Lambda$ is also continuous and therefore a distribution.

Theorem 4.20. *Let $\Lambda \in \mathcal{D}'(\mathbb{R}^d)$ and $\phi, \psi \in \mathcal{D}(\mathbb{R}^d)$. Then the following propositions hold:*

1. $T_x(\Lambda * \phi) = (T_x\Lambda * \phi) = (\Lambda * T_x\phi)$ *for all* $x \in \mathbb{R}^d$.
2. $\Lambda * \phi \in C^\infty(\mathbb{R}^d)$ *and*

   $$D^\alpha(\Lambda * \phi) = (D^\alpha\Lambda) * \phi = \Lambda * (D^\alpha\phi)$$

   for all multi-indices $\alpha \in \mathbb{N}^d$.
3. $\Lambda * (\phi * \psi) = (\Lambda * \phi) * \psi$.

Proof. For the first proposition let $x, y \in \mathbb{R}^d$. Then

$$T_x(\Lambda * \phi)(y) = (\Lambda * \phi)(y - x) = \Lambda(T_{y-x}R\phi),$$

$$(T_x\Lambda * \phi)(y) = T_x\Lambda(T_yR\phi) = \Lambda(T_{-x}T_yR\phi) = \Lambda(T_{y-x}R\phi)$$

and

$$(\Lambda * T_x\phi)(y) = \Lambda(T_yRT_x\phi) = \Lambda(T_{y-x}R\phi).$$

This proves proposition 1.

For the second proposition we note that

$$T_xR(D^\alpha\phi)(y) = R(D^\alpha\phi)(y - x) = (D^\alpha\phi)(x - y)$$

$$= (-1)^{2|\alpha|}(D^\alpha\phi)(x-y) = (-1)^{|\alpha|}D^\alpha\phi(x-y) = (-1)^{|\alpha|}D^\alpha(T_xR\phi(y))$$

for all $x, y \in \mathbb{R}^d$. Thus

$$((D^\alpha\Lambda) * \phi)(x) = D^\alpha\Lambda(T_xR\phi) = (-1)^{|\alpha|}\Lambda(D^\alpha(T_xR\phi))$$

$$= \Lambda(T_xR(D^\alpha\phi)) = (\Lambda * (D^\alpha\phi))(x),$$

proving part of the wanted equality.

Now let e_j be a vector in the standard basis of $\mathbb{R}^d$ and let $r > 0$. Denote the map $r^{-1}(T_0 - T_{re_j})$ by δ_r and note that by the first proposition we have

$$\delta_r(\Lambda * \phi) = \Lambda * \delta_r\phi.$$

Let x be a vector in $\mathbb{R}^d$, let x_j be its j:th component and define $\phi_j : \mathbb{R} \to \mathbb{K}$ with

$$\phi_j(y) = \phi(x_1, x_2, ..., x_{j-1}, y, x_{j+1}, ..., x_d).$$

Thus $D_j\phi(x) = \phi_j'(x_j)$ and by the fundamental theorem of calculus

$$r^{-1}(\phi(x) - \phi(x - re_j)) = r^{-1}\int_{x_j-h}^{x_j} \phi_j'(y)dy.$$

Note that $D^\alpha\phi$ is uniformly continuous for all $\alpha \in \mathbb{N}_0^d$. Fix $\epsilon > 0$ and let $\delta > 0$ be such that $|D_j\phi(y) - D_j\phi(x)| < \epsilon$ whenever $|y - x| < \delta$. Now

$$\Big|\delta_r\phi(x) - D_j\phi(x)\Big| = \Big|r^{-1}\int_{x_j-r}^{x_j} \phi_j'(y) - \phi_j'(x)dy\Big| \leq r^{-1}\int_{x_j-r}^{x_j} \epsilon dy = \epsilon$$

and thus $\delta_r\phi \to D_j\phi$ uniformly. Similarly $D^\alpha\delta_r\phi \to D^\alpha D_j\phi$ uniformly for every multi-index α and thus we have convergence in $\mathcal{D}(\mathbb{R}^d)$. From this we immediately have that

$$T_xR(\delta_r\phi) = \delta_r\phi(x - \cdot)$$

converges to $T_xR(D_j\phi)$ in $\mathcal{D}(\mathbb{R}^d)$ for every $x \in \mathbb{R}^d$. Therefore

$$\delta_r(\Lambda * \phi))(x) = (\Lambda * (\delta_r\phi))(x) = \Lambda(T_xR(\delta_r\phi))$$

converges to $(\Lambda * (D_j\phi))(x)$ by continuity of Λ. By iterating the above arguments we get proposition 2.

For proof of the last proposition see theorem 6.30 of [5].

$\square$

Chapter 5

Kato-Rellich theorem

5.1 Proof of the Kato-Rellich theorem

In this section we will study a situation where A is a self-adjoint or essentially self-adjoint operator and we perturb it with another operator B. The hope is that when the perturbation is small in some suitable way then self-adjointness would be preserved. This is the content of the Kato-Rellich theorem which we will present here.

First a standard result on the convergence of the Neumann series

$$\sum_{k=0}^{\infty} T^k$$

for operators in Banach spaces. Here T^0 is defined as the identity operator and $T^k = T^{k-1}T$ for $k > 0$. Note that for two bounded linear operators T and T' on a Banach space X we have

$$\|TT'x\| \leq \|T\|\|T'x\| \leq \|T\|\|T'\| \tag{5.1}$$

when $x \in X$ lies in the unit ball. Thus the product TT' is a bounded linear operator and $\|TT'\| \leq \|T\|\|T'\|$ in the operator norm.

Lemma 5.1. *Let X be a Banach space and let $T : X \to X$ be a bounded linear operator with $\|T\| < 1$. Then the Neumann series of T converges in the operator norm and*

$$(I - T)^{-1} = \sum_{k=0}^{\infty} T^k.$$

Proof. When $\|x\| \leq 1$ we have

$$\Big\| \Big(\sum_{k=0}^{n} T^k \Big) x \Big\| = \Big\| \sum_{k=0}^{n} T^k x \Big\| \leq \sum_{k=0}^{n} \|T^k x\| \leq \sum_{k=0}^{n} \|T^k\| \leq \sum_{k=0}^{n} \|T\|^k$$

by the inequality (5.1). Thus the Neumann series converges in the operator norm since $\|T\| < 1$ and the rightmost expression is a geometric series. Since the bounded operators on X form a Banach space with the operator norm, we have that $S = \lim_{k\to\infty} \sum_{k=1}^{k} T^k$ exists, and is bounded and linear as well.

Now let $S_n = \sum_{k=1}^{n} T^k$ for $n \in \mathbb{N}$. Then

$$(I - T)S_n = \sum_{k=0}^{n} T^k - \sum_{k=1}^{n+1} T^k = I - T^{n+1} = S_n(I - T)$$

and hence

$$\|(I - T)S_n - I\| = \|S_n(I - T) - I\| = \|T^{n+1}\| \leq \|T\|^{n+1}.$$

We conclude that

$$\lim_{n\to\infty} (I - T)S_n = (I - T)S = S(I - T) = I$$

as both $I - T$ and S_n are continuous for all $n \in \mathbb{N}$. □

Definition 5.2. *Let A and B be densely defined operators on a Hilbert space H. We say that B is A-bounded if $D(A) \subset D(B)$ and for some non-negative $\alpha, \beta \in \mathbb{R}$ we have*

$$\|Bx\| \leq \alpha\|Ax\| + \beta\|x\|,$$

for all $x \in D(A)$.

Theorem 5.3 (Kato-Rellich)**.** *Let A and B be densely defined symmetric operators on a Hilbert space H and let B be A-bounded with some $\alpha, \beta \in \mathbb{R}$. Then if A is self-adjoint and $\alpha < 1$ the operator $A + B : D(A) \to H$ is also self-adjoint. Also should a symmetric operator A' with $\overline{A'} = A$ exist we have that $A' + B : D(A') \to H$ is essentially self-adjoint.*

Proof. Assume first that we have proven the theorem in the case that A is self-adjoint. Then if $A = \overline{A'}$ for some symmetric operator A' we have that $\overline{A'} + B : D(\overline{A'}) \to H$ is self-adjoint. In particular $\overline{A'}+B$ is closed, as is the closure of $A'+B : D(A') \to H$. Clearly $\overline{A'}+B$ extends $A'+B$ and since it is closed we must have that $\overline{A' + B} \subset \overline{A'}+B$. Conversely let $u \in D(\overline{A'})$. Now there exists a sequence $(u_k)_{k=1}^{\infty}$ in $D(A')$ such that $(u_k, A'u_k)$ converges to $(u, \overline{A'}u)$ in $H \times H$. By assumption

$$\|B(u_k - u)\| \leq \alpha\|\overline{A'}(u_k - u)\| + \beta\|u_k - u\|$$

and thus $Bu_k \to Bu$ in H. This implies that $(u_k, (A' + B)u_k)$ converges to $(u, (\overline{A'} + B)u)$ in $H \times H$ but since $(u_k, (A'+B)u_k) \in \mathcal{G}(A'+B)$ for all $k \in \mathbb{N}$ we have that $(u, (\overline{A'}+B)u) \in \overline{\mathcal{G}(A' + B)} = \mathcal{G}(\overline{A' + B})$. Thus $\overline{A'} + B \subset \overline{A' + B}$ which implies that $A' + B$ is essentially self-adjoint on $D(A')$ and that its self-adjoint extension is $\overline{A'} + B$.

To conclude the proof assume A and B are as in definition 5.2 with $\alpha < 1$, A self-adjoint and B symmetric. We note that $A + B$ is now clearly symmetric on $D(A + B) = D(A)$.

Our aim is to show that for a suitable $\lambda > 0$ we have

$$Ran(A + B \pm i\lambda I) = H$$

since this implies that for every $h \in H$ there exists a $v \in D(A)$ such that

$$(A + B \pm i\lambda I)v = (\lambda^{-1}(A + B) \pm iI)\lambda v = h.$$

Thus $\lambda^{-1}(A + B) \pm iI$ is also a surjection and theorem 3.14 gives self-adjointness of $\lambda^{-1}(A + B)$. Since $\lambda > 0$ it follows that $A + B$ will also be self-adjoint. We will fix λ later. For now it will be some positive constant.

By assumption A is self-adjoint so its spectrum is a subset of $\mathbb{R}$ by theorem 3.19. Since $\Im(\pm i\lambda) \neq 0$ from the definition of the spectrum it follows that $(A \pm i\lambda I)^{-1}$ exists and is bounded. Also for all $u \in D(A)$ we have

$$\pm(Au|i\lambda u) = \mp(i\lambda u|Au)$$

and thus

$$\begin{aligned} \|(A \pm i\lambda I)u\|^2 &= \|Au\|^2 \pm (Au|i\lambda u) \pm (i\lambda u|Au) + \lambda^2\|u\|^2 \\ &= \|Au\|^2 + \lambda^2\|u\|^2 = \|Au\|^2 + \lambda^2\|u\|^2. \end{aligned}$$

From this and the fact that B is $A-$bounded we get

$$\|Bu\| \leq \alpha\|Au\| + \beta\|u\| \leq \alpha\|(A \pm i\lambda I)u\| + \beta\lambda^{-1}\|(A \pm i\lambda I)u\|$$

$$\leq (\alpha + \beta\lambda^{-1})\|(A \pm i\lambda I)u\|.$$

We now choose λ large enough so that $(\alpha + \beta\lambda^{-1}) < 1$. Then

$$(\alpha + \beta\lambda^{-1})\|(A \pm i\lambda I)u\| \geq \|Bu\| = \|B(A \pm i\lambda I)^{-1}(A \pm i\lambda I)u\|$$

which implies

$$(\alpha + \beta\lambda^{-1}) \geq \|B(A \pm i\lambda I)^{-1}\frac{(A \pm i\lambda I)u}{\|(A \pm i\lambda I)u\|}\|.$$

Since $(A \pm i\lambda I)$ is a bijection this implies that the operator norm of $B(A \pm i\lambda I)^{-1}$ is strictly smaller than one. By the Neumann series the operator $I + B(A \pm i\lambda I)^{-1}$ is thus invertible.

Now

$$A + B \pm i\lambda I = A \pm i\lambda I + B(A \pm i\lambda I)^{-1}(A \pm i\lambda I) = (I + B(A \pm i\lambda I)^{-1})(A \pm i\lambda I)$$

where both $(I + B(A \pm i\lambda I)^{-1})$ and $(A \pm i\lambda I)$ are bijections and so their product is also. In particular this implies that $Ran(A + B \pm i\lambda I) = H$. $\square$

5.2 Application for the Schrödinger operator

In this section we will do a case study on the Schrödinger operator which is a linear operator of the form

$$T = -\Delta + V.$$

Without giving any further information this is of course a purely formal sum. The establishing of self-adjointness of T with different assumptions on the multiplication operator V and the domain $D(T)$ has proven to be a quite rich mathematical subject. On this front T. Kato and B. Simon have been prolific contributors. Operators of this form are named after Erwin Schrödinger who first introduced his namesake equation to model the dynamics of quantum systems. The multiplication operator V above is called the potential. This comes from the usual interpretation of T as the total energy operator of a quantum system.

Here we will use the Kato-Rellich theorem to show that T is essentially self-adjoint on $C_c^\infty(\mathbb{R}^3)$, when V is a real valued element of

$$L^2(\mathbb{R}^3) + L^\infty(\mathbb{R}^3) = \{u + v : u \in L^2(\mathbb{R}^3), v \in L^\infty(\mathbb{R}^3)\}.$$

In particular this will give the essential self-adjointness of T with the Coulomb potential $V(x) = \frac{a}{|x|}$. Here a is a real constant. This is especially relevant to physics since it is basically the electric potential energy operator for a point charge.

First we will need to recall some results from Fourier analysis. We only state the theorems and refer the reader to the proofs since they are not relevant to the main topic of this thesis.

Definition 5.4. *For $f \in L^1(\mathbb{R}^d)$ we define the Fourier transform at a point $\xi \in \mathbb{R}^d$ as*

$$\hat{f}(\xi) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)dx$$

and the inverse Fourier transform as

$$\check{f}(\xi) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{ix\cdot\xi} f(x)dx.$$

Occasionally we write $\mathcal{F}f = \hat{f}$.

Both of the above integrals are well defined since $|e^{-ix}| = 1$ for all $x \in \mathbb{R}$. For the same reason $\|\hat{f}\|_\infty \leq \|f\|_{L^1}$.

Theorem 5.5. *If both f and $\hat{f}$ are in $L^1(\mathbb{R}^d)$, then $f = \check{\hat{f}}$ almost everywhere.*

Proof. See theorem 7.7 of [5]. □

Definition 5.6. *We denote by $\mathcal{S}(\mathbb{R}^d)$ the set of smooth complex valued functions f for which*

$$\|f\|_{\alpha,\beta} := \sup_{x \in \mathbb{R}^d} |x^\alpha D^\beta f| < \infty$$

for all multi-indices $\alpha, \beta \in \mathbb{N}^d$. We call $\mathcal{S}(\mathbb{R}^d)$ the space of rapidly decreasing functions.

It is quite clear that $\mathcal{D}(\mathbb{R}^d) \subset \mathcal{S}(\mathbb{R}^d)$. Similarly as before, $\|\cdot\|_{\alpha,\beta}$ form a countable collection of seminorms that make $\mathcal{S}(\mathbb{R}^d)$ into a Fréchet space[1]. The space $\mathcal{S}(\mathbb{R}^d)$ has the nice property that the Fourier transform $\mathcal{F}$, as it was defined above, is a linear homeomorphism from $\mathcal{S}(\mathbb{R}^d)$ onto itself. We will not go in to this more deeply, but [5] and [2] contain more discussion on the topic.

We define the projection map P_k on $\mathbb{R}^d$ for $k \in \{1, ..., d\}$ naturally as

$$P_k(x_1, x_2, ..., x_d) = x_k.$$

Theorem 5.7. *Suppose $f \in \mathcal{S}(\mathbb{R}^d)$, then*

$$D_k\hat{f}(\xi) = [i\widehat{P_k(\cdot)f(\cdot)}](\xi)$$

and

$$\widehat{D_k f}(\xi) = i\xi_k\hat{f}(\xi).$$

Proof. See theorem 7.4 of [5]. □

Theorem 5.8. *Let $f \in \mathcal{S}(\mathbb{R}^d)$. Then*

$$\|f\|_{L^2} = \|\hat{f}\|_{L^2}.$$

Proof. See theorem 7.9 of [5]. □

This is of course the Plancherel theorem, although usually it is stated for the whole of $L^2(\mathbb{R}^d)$. For us this less general version suffices.

Now we are ready to start proving the main theorem of this section. First a lemma.

[1] See theorem V.9 of [2]

Lemma 5.9. *Let $a > 0$ and let $\phi \in C_c^\infty(\mathbb{R}^3)$. Then there exists a number b that is independent of ϕ such that*

$$\|\phi\|_\infty \leq a\| - \Delta\phi\|_{L^2} + b\|\phi\|_{L^2}.$$

Proof. Let $a > 0$ and fix $\phi \in C_c^\infty(\mathbb{R}^3)$. By the Schwartz inequality

$$\|\hat{\phi}\|_{L^1} \leq \|(1 + |\cdot|^2)^{-1}\|_{L^2}\|(1 + |\cdot|^2)\hat{\phi}\|_{L^2} \leq \|(1 + |\cdot|^2)^{-1}\|_{L^2}(\|\hat{\phi}\|_{L^2} + \||\cdot|^2\hat{\phi}\|_{L^2})$$

where the last term is finite since $\phi, \Delta\phi \in C_c^\infty(\mathbb{R}^3) \subset \mathcal{S}(\mathbb{R}^3)$ and by theorems 5.7 and 5.8 we have

$$\infty > \|\Delta\phi\|_{L^2} = \|\widehat{\Delta\phi}\|_{L^2} = \||\cdot|^2\hat{\phi}\|_{L^2}.$$

The function $f(x) = (1 + |x|^2)^{-1}$ is also square integrable. To see this we define the sets

$$B_N = B(0, N) \setminus B(0, N - 1)$$

for $N \in \mathbb{N}$ and note that

$$\int_{\mathbb{R}^3} |f(x)|^2 dx = \sum_{N=1}^{\infty} \int_{B_N} |f(x)|^2 dx \leq \sum_{N=1}^{\infty} \int_{B_N} |f(x)|^2 dx$$

$$\leq \int_{B(0,1)} |f(x)|^2 dx + \sum_{N=2}^{\infty} \int_{B_N} \frac{1}{|x|^4} dx.$$

Now the integral of $|f|^2$ over the unit ball is clearly finite as f is continuous. Notice that

$$\int_{B_N} |x|^{-4} dx \leq \frac{1}{(N-1)^4} \int_{B_N} dx = \frac{V(B(0,N)) - V(B(0,N-1))}{(N-1)^4}$$

where $V(B(0,r)) = \frac{4\pi}{3}r^3$, i.e., the volume of a ball of radius r in $\mathbb{R}^3$. Since

$$\frac{V(B(0,N)) - V(B(0,N-1))}{(N-1)^4} = (\frac{4\pi}{3})\frac{3N^2 - 3N + 1}{(N-1)^4} \leq \frac{\pi}{2N^2},$$

we have that $\sum_{N=2}^{\infty} \int_{B_N} |x|^{-4}$ converges.

We denote $\|f\|_{L^2} = c$ from now on. Thus far we have proven that $\hat{\phi} \in L^1(\mathbb{R}^3)$ and

$$\|\hat{\phi}\|_{L^1} \leq c(\|\hat{\phi}\|_{L^2} + \||\cdot|^2\hat{\phi}\|_{L^2}).$$

Now for any $t > 0$ we define $\hat{\phi}_t(\xi) = t^3\hat{\phi}(t\xi)$. Note that after the variable change $\xi \mapsto t^{-1}\xi$ we have

$$\int_{\mathbb{R}^3} t^3\hat{\phi}(t\xi)d\xi = \int_{\mathbb{R}^3} \hat{\phi}(\xi)d\xi,$$

since the determinant of the Jacobian of the map $r^{-1}\xi$ is t^{-3} in $\mathbb{R}^3$. Thus $\|\hat{\phi}_t\|_{L^1} = \|\hat{\phi}\|_{L^1}$ and with the same change of variables $\|\hat{\phi}_t\|_{L^2} = t^{\frac{3}{2}}\|\hat{\phi}\|_{L^2}$ and $\||\cdot|^2\hat{\phi}_t\|_{L^2} = t^{-\frac{1}{2}}\||\cdot|^2\hat{\phi}\|_{L^2}$. Hence,

$$\|\hat{\phi}\|_{L^1} \leq ct^{\frac{3}{2}}\|\hat{\phi}\|_{L^2} + ct^{-\frac{1}{2}}\||\cdot|^2\hat{\phi}\|_{L^2} = ct^{\frac{3}{2}}\|\phi\|_{L^2} + ct^{-\frac{1}{2}}\|\Delta\phi\|_{L^2}$$

by the Plancherel theorem. By theorem 5.5 we have

$$\|\hat{\phi}\|_{L^1} = \|\mathcal{F}\mathcal{F}^{-1}\hat{\phi}\|_{L^1} \geq \|\check{\hat{\phi}}\|_\infty = \|\phi\|_\infty$$

and the claim follows when we let $t = (\frac{c}{a})^2$. □

The proof of our next theorem uses the fact that $-\Delta$ is essentially self-adjoint as an operator on $L^2(\mathbb{R}^3)$ with the domain $C_c^\infty(\mathbb{R}^3)$. A priori we do not know this, but it will be a consequence of theorem 7.2 so we will take it as given for now.

Theorem 5.10. *Suppose $V = V_1 + V_2$ where $V_1 \in L^2(\mathbb{R}^3)$ and $V_2 \in L^\infty(\mathbb{R}^3)$ are real valued. Then the operator $T = -\Delta + V$ is essentially self-adjoint on $L^2(\mathbb{R}^3)$ with the domain $D(T) = C_c^\infty(\mathbb{R}^3)$.*

Proof. Recall from example 3.12 that the multiplication operator V is self-adjoint on the domain

$$D(V) = \{\phi \in L^2(\mathbb{R}^3) : V\phi \in L^2(\mathbb{R}^3)\}.$$

If $\phi \in C_c^\infty(\mathbb{R}^3)$ we have

$$\|V\phi\|_{L^2} \leq \|V_1\phi\|_{L^2} + \|V_2\phi\|_{L^2} \leq \|V_1\|_{L^2}\|\phi\|_\infty + \|V_2\|_\infty\|\phi\|_{L^2}$$

and thus $\phi \in D(V)$. For any $a > 0$ lemma 5.9 gives a number $b >$ such that

$$\|\phi\|_\infty \leq a\| - \Delta\phi\|_{L^2} + b\|\phi\|_{L^2}$$

and hence

$$\|V\phi\|_{L^2} \leq a\|V_1\|_{L^2}\| - \Delta\phi\|_{L^2} + (b\|V_1\|_{L^2} + \|V_2\|_\infty)\|\phi\|_{L^2}.$$

Now $D(-\Delta) = C_c^\infty(\mathbb{R}^3) \subset D(V)$ and the operator V is $-\Delta$ bounded with an arbitrarily small bound a. Thus the Kato-Rellich theorem implies that T is essentially self-adjoint on $C_c^\infty(\mathbb{R}^3)$. □

Chapter 6

Mollifiers and Kato's inequality

6.1 Approximate identities and convolution

In this chapter we gather some mollification methods that will be useful later.

Definition 6.1. *A collection of functions $\{\eta_\delta\}_{\delta>0}$ in $C_c^\infty(\mathbb{R}^d)$ is called a smooth approximate identity when*

1. *$\eta_\delta(x) \geq 0$ for all $x \in \mathbb{R}^d$*
2. *$supp(\eta_\delta) \subset \overline{B}(0, \frac{1}{\delta})$*
3. *for each δ we have $\int_{\mathbb{R}^d} \eta_\delta dx = 1$.*

These functions are also sometimes called mollifiers. To make sure that the set of approximate identities in $C_c^\infty(\mathbb{R}^d)$ is not empty we better present an example.

Example 6.2. For $x \in \mathbb{R}^d$ define

$$\eta_0(x) = \begin{cases} e^{\frac{1}{|x|^2-1}} & |x| < 1 \\ 0 & |x| \geq 1. \end{cases}$$

By scaling with a proper constant $C > 0$ we can ensure that the integral of $C\eta_0$ equals one. Notice that η_0 is compactly supported and continuous so it is indeed integrable. Let us denote this rescaled map with η. Now if for every $\delta > 0$ we define $\eta_\delta : \mathbb{R}^d \to \mathbb{R}$ as $\eta_\delta(x) = \delta^d \eta(\delta x)$ we will find that the maps η_δ form a smooth approximate identity. The smoothness of η is a standard proof by induction.

Next we introduce some properties of mollifiers that will be used later on. First a definition.

Definition 6.3. *Let $\{\Lambda_k\}_{k=1}^{\infty}$ be a sequence in the distribution space $\mathcal{D}'(\Omega)$. We define*

$$\lim_{k\to\infty} \Lambda_k = \Lambda \ in\ \mathcal{D}'(\Omega)$$

to mean that

$$\lim_{k\to\infty} \Lambda_k\phi = \Lambda\phi$$

for every $\phi \in \mathcal{D}(\Omega)$.

Theorem 6.4. *Let $\Lambda \in \mathcal{D}'(\mathbb{R}^d)$ and η_δ as in definition 6.1. Then*

$$\Lambda * \eta_\delta \to \Lambda \ as\ \delta \to \infty \ in\ \mathcal{D}'(\mathbb{R}^d).$$

Proof. First let f be a $C(\mathbb{R}^d)$ function and let $K \subset \mathbb{R}^d$ be compact. Then $f * \eta_\delta$ converges to f uniformly on K. This is a consequence of the estimate

$$|(f*\eta)(x)-f(x)| = \Big|\int_{\mathbb{R}^d} f(y)\eta_\delta(x-y)dy - \int_{\mathbb{R}^d} f(x)\eta_\delta(x-y)dy\Big| \le \int_{\mathbb{R}^d} \eta_\delta(x-y)|f(x)-f(y)|dy$$

$$= \int_{B(x,\frac{1}{\delta})} \eta_\delta(x-y)|f(x)-f(y)|dy.$$

Uniform convergence follows from the uniform continuity of f on K and the properties of η_δ.

Next let $\phi \in \mathcal{D}(\mathbb{R}^d)$ and note that $\phi * \eta_\delta$ is compactly supported since

$$\phi(y)T_xR\eta_\delta(y)$$

is zero as a function of y outside of the set $K_1 \cap (x - K_2)$ where K_1 and K_2 are the supports of ϕ and η_δ respectively. Now $D^\alpha(\phi * \eta_\delta) = (D^\alpha\phi) * \eta_\delta$ converges to $D^\alpha\phi$ uniformly for all α since ϕ is smooth. Thus $\phi * \eta_\delta$ converges to ϕ in $\mathcal{D}(\mathbb{R}^d)$.

The result now follows from theorem 4.20 since

$$\Lambda(R\phi) = \Lambda(R \lim_{\delta\to\infty}(\eta_\delta * \phi)) = \lim_{\delta\to\infty} \Lambda(R(\eta_\delta * \phi)) = \lim_{\delta\to\infty}(\Lambda * (\eta_\delta * \phi))(0)$$

$$= \lim_{\delta\to\infty}((\Lambda * \eta_\delta) * \phi)(0) = \lim_{\delta\to\infty}(\Lambda * \eta_\delta)(R\phi).$$

By substituting ϕ with $\phi(-\cdot)$ we get

$$\Lambda(\phi) = \lim_{\delta\to\infty}(\Lambda * \eta_\delta)(\phi).$$

□

Theorem 6.5. *Let $u \in L^p(\mathbb{R}^d)$ for some $p \in [1,\infty)$ and $\phi \in C_c^\infty(\mathbb{R}^d)$. Then $u * \phi \in L^p(\mathbb{R}^d)$.*

Proof. Let $q = \frac{p}{p-1}$ and $supp(\phi) \subset K$ where $K \in \mathbb{R}^d$ is compact. First by the use of Hölder's inequality we evaluate

$$|\phi * u| = |\int_{\mathbb{R}^d} \phi(x-y)u(y)dx| \leq \int_{\mathbb{R}^d} |\phi(x-y)u(y)|dx$$

$$= \int_{\mathbb{R}^d} |\phi(x-y)|^{\frac{1}{q}}|\phi(x-y)|^{\frac{1}{p}}|u(y)|dy \leq \Big(\int_{\mathbb{R}^d} |\phi(x-y)|dy\Big)^{\frac{1}{q}}\Big(\int_{\mathbb{R}^d} |\phi(x-y)||u(y)|^p dy\Big)^{\frac{1}{p}}$$

$$\leq \Big(m(K)\|\phi\|_\infty\Big)^{\frac{1}{q}}\Big(\int_{\mathbb{R}^d} |\phi(x-y)||u(y)|^p dy\Big)^{\frac{1}{p}}.$$

Where m is the usual Lebesgue measure on $\mathbb{R}^d$. We denote $m(K)\|\phi\|_\infty = C$ which is of course finite. Now by the above calculation and Tonelli's theorem

$$\|\phi * u\|_p^p = \int_{\mathbb{R}^d} |\phi * u|^p dx \leq C^{\frac{p}{q}} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |\phi(x-y)||u(y)|^p dydx$$

$$= C^{\frac{p}{q}} \int_{\mathbb{R}^d} |u(y)|^p \int_{\mathbb{R}^d} |\phi(x-y)|dxdy \leq C^{\frac{p}{q}+1}\|u\|_p^p.$$

Thus $u * \phi = \phi * u \in L^p(\mathbb{R}^d)$. □

Theorem 6.6. *Let η_δ be as in example 6.2 and $u \in L^p_{loc}(\mathbb{R}^d)$ with $1 \leq p < \infty$. Then $u * \eta_\delta \to u$ pointwise almost everywhere.*

Proof. We make use of the Lebesgue differentiation theorem here which says that for $f \in L^1(\mathbb{R}^d)$ and almost every $x \in \mathbb{R}^d$ we have

$$\lim_{r\to 0} \frac{1}{m(B(x,r))}\int_{B(x,r)} |f(x) - f(y)|dy = 0,$$

where $B(x,r)$ is an open ball of radius $r > 0$ with center at point x. These points of convergence are called Lebesgue points of the function f. The proof and further discussion on the topic can be found from chapter seven of [7].

Here we only assume $u \in L^p_{loc}(\mathbb{R}^d)$ but for every $k \in \mathbb{N}$ we have $\chi_{B(0,k)}u \in L^1(\mathbb{R}^d)$ where χ is the characteristic function. The the Lebesgue theorem now holds for $\chi_{B(0,k)}u$, so this causes no trouble for us.

Now let $x \in \mathbb{R}^d$ be a Lebesgue point of u. With a change of variables and the properties of η_δ we get

$$|u(x) - u*\eta_\delta(x)| = |u(x)\int_{\mathbb{R}^d} \eta_\delta(x-y)dy - \int_{\mathbb{R}^d} \eta_\delta(x-y)u(y)dy| \leq \int_{\mathbb{R}^d} \eta_\delta(x-y)|u(x)-u(y)|dy$$

$$= \delta^d \int_{B(x,\delta^{-1})} \eta(\delta|x-y|)|u(x)-u(y)|dy \leq \|\eta\|_\infty \delta^d \int_{B(x,\delta^{-1})} |u(x)-u(y)|dy$$
$$= \|\eta\|_\infty \frac{c_d}{m(B(x,\delta^{-1}))} \int_{B(x,\delta^{-1})} |u(x)-u(y)|dy,$$

where c_d is the Lebesgue measure of the unit ball in $\mathbb{R}^d$. Thus by taking limits with $\delta \to \infty$ the Lebesgue differentiation theorem gives that $u * \eta_\delta$ converges to u at every Lebesgue point of u. The result follows since almost every element of $\mathbb{R}^d$ is a Lebesgue point of u.

□

Theorem 6.7. *Let $f \in L^p(\mathbb{R}^d)$ with $1 \leq p < \infty$. Then*

$$\|f * \eta_\delta - f\|_{L^p} \to 0$$

as $\delta \to \infty$.

Proof. See theorem 9.6 and corollary 9.7 of [10]. □

6.2 Kato's inequality

Let T and S be distributions. We define $T \leq S \Leftrightarrow S \geq T$ to mean that $T\phi \leq S\phi$ for all test functions ϕ that are non-negative pointwise.

The next lemma is critical in establishing the main result in the next chapter and also for other similar self-adjointness results[1].

Lemma 6.8 (Kato's inequality)**.** *Suppose that u and Δu are both elements of $L^1_{loc}(\mathbb{R}^d)$. Denote*

$$sgn(u) = \begin{cases} \frac{\overline{u}}{|u|} & u \neq 0 \\ 0 & u = 0. \end{cases}$$

Then

$$\Delta|u| \geq \Re(sgn(u)\Delta u) \tag{6.1}$$

in the distributional sense.

[1]See section X.4 of [3].

Proof. Suppose first that $u \in C^\infty(\mathbb{R}^d)$ and define $u_\epsilon = \sqrt{|u|^2 + \epsilon}$ where $\epsilon > 0$. By induction it is a simple task to show that for any multi-index α it holds that $D^\alpha u_\epsilon$ is of the from $\sum_{k=1}^n P_k u_\epsilon^{-j_k}$ where $P_1, ..., P_n \in C^\infty(\mathbb{R}^d)$ and $n, j_1, ..., j_n \in \mathbb{N}$. Now since u_ϵ^{-j} is continuous for every $j \in \mathbb{N}$ we see that u_ϵ is smooth.

Let $1 \leq j \leq d$. Then we have $2u_\epsilon \partial_j u_\epsilon = \Re(2\overline{u}\partial_j u)$ since

$$2\overline{u}\partial_j u = 2(\Re(u) - i\Im(u))(\partial_j \Re(u) + i\partial_j \Im(u))$$

$$= 2\Re(u)\partial_j\Re(u) + 2\Im(u)\partial_j\Im(u) + i(2\Re(u)\partial_j\Im(u) - 2\Im(u)\partial_j\Re(u))$$

and

$$\partial_j(|u|^2 + \epsilon) = \partial_j(\Re(u)^2 + \Im(u)^2 + \epsilon) = 2\Re(u)\partial_j\Re(u) + 2\Im(u)\partial_j\Im(u).$$

So we get the equality

$$u_\epsilon \nabla u_\epsilon = \Re(\overline{u}\nabla u). \tag{6.2}$$

Here the right-hand side is to be understood as a vector whose components consist of the real parts of the components of $\overline{u}\nabla u$. Now if we take the divergence on both sides we end up with

$$u_\epsilon \Delta u_\epsilon + |\nabla u_\epsilon|^2 = \Re(\overline{u}\Delta u) + |\nabla u|^2. \tag{6.3}$$

From equality (6.2) and the fact that $|u_\epsilon| \geq |u|$ we can make the approximation

$$|\nabla u_\epsilon| = |u_\epsilon|^{-1}|\Re(\overline{u}\nabla u)| \leq |u_\epsilon|^{-1}|\overline{u}||\nabla u| \leq |\nabla u|$$

which implies with (6.3) that

$$u_\epsilon \Delta u_\epsilon \geq \Re(\overline{u}\Delta u).$$

Dividing by u_ϵ we arrive at the inequality

$$\Delta u_\epsilon \geq \Re(\frac{\overline{u}}{u_\epsilon}\Delta u). \tag{6.4}$$

This holds pointwise so it also holds for the distributions induced by the locally integrable functions Δu_ϵ and $\Re(\frac{\overline{u}}{u_\epsilon}\Delta u)$.

Suppose now that u and Δu are both in $L^1_{loc}(\mathbb{R}^d)$. Let η_δ be a smooth approximate identity as in example 6.2 and define $u^\delta = u * \eta_\delta$. Now $u^\delta \in C^\infty(\mathbb{R}^d)$ by theorem 4.20 and thus the inequality (6.4) holds for u^δ with every $\delta, \epsilon > 0$. By theorems 6.4 and 6.6 the functions u^δ converge to u in $D'(\mathbb{R}^d)$ and also pointwise almost everywhere. Since

$$sgn_\epsilon(z) := \frac{\overline{z}}{\sqrt{|z|^2 + \epsilon}}$$

is a continuous function in z for every $\epsilon > 0$, it follows that

$$sgn_\epsilon(u^\delta) \to sgn_\epsilon(u)$$

pointwise almost everywhere.

By theorem 4.20

$$\Delta u^\delta = \Delta(u * \eta_\delta) = (\Delta u * \eta_\delta) = (\Delta u)^\delta$$

and so we have that the maps Δu^δ converge to Δu in $D'(\mathbb{R}^d)$.

Next we need to show that $sgn_\epsilon(u^\delta)\Delta u^\delta \to sgn_\epsilon(u)\Delta u$ in $D'(\mathbb{R}^d)$ which implies that the inequality (6.4) holds for u. To this end, let ϕ be some test function with support in a compact set $K \subset \mathbb{R}^d$ and let $R > 0$ be such that $B(0,R)$ contains K. Now

$$|\int_{\mathbb{R}^d} \phi sgn_\epsilon(u^\delta)\Delta u^\delta dx - \int_{\mathbb{R}^d} \phi sgn_\epsilon(u)\Delta u dx|$$

$$= |\int_{\mathbb{R}^d} \phi(sgn_\epsilon(u^\delta)\Delta u^\delta - sgn_\epsilon(u^\delta)\Delta u)dx + \int_{\mathbb{R}^d} \phi(sgn_\epsilon(u^\delta)\Delta u - sgn_\epsilon(u)\Delta u)dx|$$

$$\leq |\int_{\mathbb{R}^d} \phi sgn_\epsilon(u^\delta)(\Delta u^\delta - \Delta u)dx| + |\int_{\mathbb{R}^d} \phi \Delta u(sgn_\epsilon(u^\delta) - sgn_\epsilon(u))dx|.$$

We will show that both of the terms in the last expression converge to zero.

First note that

$$|\int_{\mathbb{R}^d} \phi sgn_\epsilon(u^\delta)(\Delta u^\delta - \Delta u)dx| \leq \|\phi\|_\infty \int_{B(0,R)} |\Delta u^\delta - \Delta u| dx.$$

By the fact that $B(0,R)$ is a balanced set, we have

$$\chi_{B(0,R)}(x-y) = 1 \Leftrightarrow x - y \in B(0,R) \Leftrightarrow y - x \in B(0,R) \Leftrightarrow y \in B(x,R)$$

for all $x - y \in \mathbb{R}^d$. Here χ_S is the indicator function of a set S. Now

$$(\chi_{B(0,2R)}\Delta u * \eta_\delta)|_{B(0,R)}(x) = \int_{B(0,\delta^{-1})} \chi_{B(x,2R)}(y)\chi_{B(0,R)}(x)\Delta u(x-y)\eta_\delta(y)dy$$

$$= \int_{B(0,\delta^{-1})} \chi_{B(0,R)}(x)\Delta u(x-y)\eta_\delta(y)dy = (\Delta u * \eta_\delta)|_{B(0,R)}(x)$$

whenever $\delta^{-1} < R$ since then $B(0,\delta^{-1}) \subset B(x,2R)$. Thus

$$\int_{B(0,R)} |\Delta u^\delta - \Delta u| dx = \int_{B(0,R)} |((\chi_{B(0,2R)}\Delta u) * \eta_\delta) - \Delta u| dx$$

$$\leq \int_{B(0,2R)} |((\chi_{B(0,2R)}\Delta u) * \eta_\delta) - \Delta u| dx$$

$$\leq \int_{\mathbb{R}^d} |((\chi_{B(0,2R)}\Delta u) * \eta_\delta) - \chi_{B(0,2R)}\Delta u| dx \to 0, \text{ as } \delta \to \infty.$$

Here we used theorem 6.7.

For the second term we use dominated convergence. We already know that

$$|\phi\Delta u||sgn_\epsilon(u^\delta) - sgn_\epsilon(u)|$$

converges to 0 pointwise almost everywhere and also

$$|\phi\Delta u||sgn_\epsilon(u^\delta) - sgn_\epsilon(u)| \leq 2|\phi\Delta u| \in L^1(\mathbb{R}^d).$$

By the dominated convergence theorem

$$\lim_{\delta\to\infty} |\int_{\mathbb{R}^d} \phi\Delta u(sgn_\epsilon(u^\delta) - sgn_\epsilon(u))dx| \leq \lim_{\delta\to\infty} \int_{\mathbb{R}^d} |\phi\Delta u||sgn_\epsilon(u^\delta) - sgn_\epsilon(u)|dx$$

$$= \int_{\mathbb{R}^d} |\phi\Delta u| \lim_{\delta\to\infty} |sgn_\epsilon(u^\delta) - sgn_\epsilon(u)|dx = 0.$$

We have now proven that $\Delta u_\epsilon \geq \Re(sgn_\epsilon(u)\Delta u)$ holds for all $\epsilon > 0$.

Finally we let $\epsilon \to 0$. Since $sgn_\epsilon(u) \to sgn(u)$ pointwise and

$$|\Re(sgn_\epsilon(u)\Delta u)| \leq |\Delta u|,$$

by dominated convergence we have that $\Re(sgn_\epsilon(u)\Delta u)$ converges to the right-hand side of inequality (6.1) in $D'(\mathbb{R}^d)$. For the left-hand side let ϕ be a test function. Then, as u_ϵ is clearly locally integrable and thus a distribution, and since $u_\epsilon \geq |u|$ pointwise we have

$$\Big|\int_{\mathbb{R}^d} \phi\Delta u_\epsilon dx - \int_{\mathbb{R}^d} \phi\Delta|u|dx\Big| = \Big|\int_{\mathbb{R}^d} \phi(\Delta u_\epsilon - \Delta|u|)dx\Big| = \Big|\int_{\mathbb{R}^d} (\Delta\phi)(u_\epsilon - |u|)dx\Big|$$

$$\leq \int_{\mathbb{R}^d} |\Delta\phi|(u_\epsilon - |u|)dx \leq \sqrt{\epsilon}\int_{\mathbb{R}^d} |\Delta\phi|dx.$$

Thus Δu_ϵ converges to $\Delta|u|$ in the distributional sense and so the inequality (6.1) holds.

□

Chapter 7

Schrödinger operators with positive potentials

Usually the dimension of the underlying euclidean space plays a crucial role[1] in establishing the self-adjointness of the operator T introduced in chapter 5. Here we will present a result first proven by Kato in [4] and then Simon in [9] which states that T is essentially self-adjoint on $C_c^\infty(\mathbb{R}^d)$ with the relatively general assumption that V be pointwise non-negative and locally square-integrable with no restriction on the dimension d.

First we show that $-\Delta$ is non-negative as an operator.

Lemma 7.1. *Let $\phi \in C_c^\infty(\mathbb{R}^d)$. Then $(-\Delta\phi|\phi) \geq 0$.*

Proof. Fix $\phi \in C_c^\infty(\mathbb{R}^d)$ and let $B \subset \mathbb{R}^d$ be an open ball that contains the support of ϕ. Using Green's first identity we get

$$\int_{\mathbb{R}^d} \overline{\phi}\Delta\phi + \nabla\overline{\phi} \cdot \nabla\phi dx = \int_B \overline{\phi}\Delta\phi + \nabla\overline{\phi} \cdot \nabla\phi dx = \int_{\partial B} \overline{\phi}(\nabla\phi \cdot \mathbf{n}) dS = 0.$$

Since the domain of ϕ is $\mathbb{R}^d$, differentiation commutes with complex conjugation and we have that

$$(-\Delta\phi|\phi) = -\int_{\mathbb{R}^d} \overline{\phi}\Delta\phi dx = \int_{\mathbb{R}^d} \nabla\overline{\phi} \cdot \nabla\phi dx = \int_{\mathbb{R}^d} |\nabla\phi|^2 \geq 0.$$

□

Note that we have $(-\Delta\phi|\phi) = 0$ if and only if ϕ is identically zero. This is because ϕ was compactly supported so the norm of the gradient can only be zero if the whole function is.

Finally we are ready to prove the general essential self-adjointness result for positive potentials.

[1] Theorem 5.10 for example.

Theorem 7.2. *Let $V \in L^2_{loc}(\mathbb{R}^d)$ and $V \geq 0$ pointwise almost everywhere. Then the operator $T := -\Delta + V$ is essentially self-adjoint on $L^2(\mathbb{R}^d)$ with domain $D(T) = C_c^\infty(\mathbb{R}^d)$.*

Proof. Define $A := -\Delta + V + I$. It is clear that essential self-adjointness of the operator A in $D(T)$ is equivalent to essential self-adjointness of T in $D(T)$. The operator A is now symmetric since each term is symmetric, and strictly positive because

$$(A\phi|\phi) = \int_{\mathbb{R}^d} (A\phi)\overline{\phi}dx = -\int_{\mathbb{R}^d} (\Delta\phi)\overline{\phi}dx + \int_{\mathbb{R}^d} V|\phi|^2 dx + \int_{\mathbb{R}^d} \phi\overline{\phi} \geq (\phi|\phi),$$

for all $\phi \in D(T)$. The inequality above follows since the first term in the preceding sum is positive by Lemma 7.1 and the second term is positive because V was assumed to be pointwise non-negative. Thus, it is enough to prove that $Ker(A^*) = \{0\}$ by theorem 3.16.

For this, assume that $A^*u = 0$ for some $u \in D(A^*)$. Since the domain of T, and consequently also of A, is chosen as $C_0^\infty(\mathbb{R}^d)$, it follows from the definition of the adjoint that $A^*u = 0$ implies $(-\Delta_w + V + I)u = 0$. Here Δ_w is the distributional Laplacian. This comes from the fact that for all $v \in C_c^\infty(\mathbb{R}^d)$, we have

$$(Av|u) = (v|A^*u) \Leftrightarrow (-\Delta v|u) = -\int_{\mathbb{R}^d} v((V+I)\overline{u} - \overline{A^*u})dx.$$

Thus we have that

$$(-\Delta_w + V + I)u = 0,$$

which implies $\Delta_w u \in L^1_{loc}(\mathbb{R}^d)$, since $u \in L^2(\mathbb{R}^d) \subset L^2_{loc}(\mathbb{R}^d) \subset L^1_{loc}(\mathbb{R}^d)$ and $Vu \in L^1_{loc}(\mathbb{R}^d)$ by the Schwartz inequality. From now on we will just denote Δ_w by Δ, since the meaning will be clear from context.

We can now utilize the distributional inequality from lemma 6.8 to get

$$\Delta|u| \geq \Re(sgn(u)\Delta u) = \Re(sgn(u)(Vu + u))) = V|u| + |u|,$$

since

$$sgn(u)u = \frac{\overline{u}u}{|u|} = |u| \text{ when } u \neq 0.$$

In particular, this implies that $\Delta|u|$ is non-negative in the distributional sense.

We now make use of the approximate identity η_δ from definition 6.1. If we denote $w = |u|$ and $w^\delta = w * \eta_\delta$, then $w^\delta \in C^\infty(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$,

$$\Delta w^\delta = w * \Delta\eta_\delta = \Delta w * \eta_\delta$$

and $\Delta w^\delta \in L^2(\mathbb{R}^d)$ by theorems 4.20 and 6.5.

Since $\Delta|u|$ is distributionally non-negative and η_δ is smooth, compactly supported and non-negative

$$\Delta w^\delta(x) = \int_{\mathbb{R}^d} \Delta|u(x-y)|\eta_\delta(y)dy \geq 0.$$

Therefore the function Δw^δ is pointwise non-negative and since w^δ is also pointwise non-negative, we have $(w^\delta, \Delta w^\delta) \geq 0$.

On the other hand, let χ_{B_k} be the characteristic function of a ball with radius $k \in \mathbb{N}$ centered at the origin. Denote by w_k^δ the function $(\chi_{B_k} w) * \eta_\delta$. Now, if $K \subset \mathbb{R}^d$ is a compact set containing the support of η_δ we have

$$w_k^\delta(x) = \int_{\mathbb{R}^d} \chi_{B_k}(x-y)|u(x-y)|\eta_\delta(y)dy = \int_{B(x,k)} |u(x-y)|\eta_\delta(y)dy$$

$$= \int_{B(x,k)\cap K} |u(x-y)|\eta_\delta(y)dy$$

for all $x \in \mathbb{R}^d$, $k > 0$ and $\delta > 0$. Thus w_k^δ is a compactly supported smooth function, with support in the set $\{x \in \mathbb{R}^d : d(x,K) \leq 2k\}$. The functions w_k^δ and $\Delta w_k^\delta = (\chi_{B_k} w) * \Delta\eta_\delta$ both converge pointwise to w^δ and Δw^δ respectively as $k \to \infty$, with

$$|w_k^\delta(x)| \leq w^\delta(x)$$

and

$$|\Delta w_k^\delta(x)| \leq \Delta w^\delta(x).$$

By the dominated convergence theorem we have convergence also in the L^2 sense. Thus,

$$(w_k^\delta|\Delta w_k^\delta) \to (w^\delta|\Delta w^\delta)$$

and since $(w_k^\delta|\Delta w_k^\delta) \leq 0$ by lemma 7.1, we also have $(w^\delta|\Delta w^\delta) \leq 0$.

We have proven that $(\Delta w^\delta|w^\delta) = 0$, which implies that $w^\delta = 0$. Since $w^\delta \to w$ pointwise almost everywhere, we also have $w = |u| = 0$. Thus $u = 0$, which concludes the proof. $\square$

From this result we get quite a large family of potentials for which the Schrödinger operator is essentially self-adjoint on $C_c^\infty(\mathbb{R}^d)$. For instance any potential $V \in L^2_{loc}(\mathbb{R}^d)$ that is bounded from below will do, since a constant shift preserves self-adjointness. Additionally every continuous function that is bounded from below will do as well. For example this includes all polynomials with even powers.

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